A Course in Arithmetic

COURSE IN ARITHMETIC

A TREATISE IN THREE PARTS

COMPLETE IN ONE VOLUME

BY

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PREFACE.

In presenting to the public a new treatise on Arithmetic, it is proper to call attention to the features which distinguish it from others, and to the need of radical changes in the traditional precepts retained in the text-books in current use.

First, it is intended to present, in the compass of a single volume, a complete treatise, and, indeed, all that the average pupil needs to study on the subject from first to last. It is supposed the pupil has previously acquired those first elementary notions of numbers, such as may be best taught orally, and, indeed, should be learned before he is prepared to use a book for systematic study.

Secondly, the subject-matter is arranged in three parts logically distinct, in each of which the pupil gives his attention to a specific object, complete in itself.

In the first part the *simple operations* are explained, and exercises are given for practice. Here the pupil learns how to add, subtract, multiply, and divide numbers, both integral and fractional.

Exercises are given in each of the operations, but no problems, and upon reaching the conclusion of this part the student should be able to perform, with readiness and accuracy, any one of the fundamental operations upon any numbers, whole or fractional, whether of the common or decimal form.

In actual experience, numbers are chiefly used in the measures of quantities, as of weights, of lengths, of surfaces, of time, and

of various other kinds. These measures form the material with which the arithmetical workman has to deal. The next step, then, is to become acquainted with this material, and in the second part of the treatise will be found an explanation of the various measures in use, including the metric system. Exercises in reducing measures expressed in given denominations to equivalents expressed in others, furnish pleasant and profitable practice for the learner, and prepare the way for the third part of the course.

Having reached this point, with a thorough knowledge of what precedes, the student is well prepared to undertake in the third part the various applications of numerical computation to practical questions, such as problems of ratio, proportion, percentage in its various forms, square and cube roots, and miscellaneous problems.

In the details of the third part it is thought better that the student should become thoroughly familiar with general principles rather than to occupy time with every elaborate modification that can be devised.

The foregoing briefly delineates the arrangement of the subject-matter in the following treatise.

A third marked feature will be found in the character of the definitions and of the expositions of principles. The treatment of each topic is intended to be lucid and logical, and at the same time to be in harmony with rigorous scientific accuracy. In carrying out this plan I have found it necessary to discard many of the time-honored phrases, which have been accepted as inevitable heirlooms, also to omit many superfluities, and in many instances, to modify in important particulars the expositions of principles. Thus it is frequently stated that "number is a unit or a collection of units;" that "a unit is a single thing, as an orange or a horse." Hence, it would follow, a number may have four legs and be able to run.

Properly speaking, a number may express (or connote) a unit or a collection of units, but it can neither be a unit nor a collection of units.

Again, numbers are usually described as *abstract* or *concrete*. But it seems clear that all numbers are abstract, and in their nature can never be concrete, and it can be easily seen that the use of the phrase, "concrete number," tends to the confounding of number with the things numbered.

An examination of the ordinary definitions of so simple a process as addition will show them to be very imperfect, and the same may be said of multiplication and of division; and it is believed, particularly in the case of division, that an exposition of the process, based upon a true definition, is much more easily comprehended by the learner.

Again, in the case of fractional numbers, the distinction between the number and the symbol that represents it seems to be often overlooked. In this treatise the term *fraction* is applied only to the symbol, and not to the number itself. It will no doubt be admitted that an expression commonly called a fraction may perform either one of two offices. For instance, the expression $\frac{40}{13}$ may indicate that 40 is to be divided by 13, or it may express 40 thirteenths. It is of course true that the numerical results are equivalent, but it will scarcely be denied that the processes suggested in the mind are different, and that a recognition of this distinction must tend to simplify the presentation of the subject.

Indeed, it will in general be found that a true definition, or a true exposition of a process, will not only be more easily understood, but will tend to lessen the difficulty of subsequent acquisitions and make them more agreeable.

It will be noticed that the metric system is given an equal prominence with the common systems of measures, and this, probably, needs no defense.

It will also be noticed that answers to many of the problems are not given. This omission is dictated by a conviction resulting from long experience in teaching that in this way the pupil will gain in self-reliance, as he must of necessity use more thorough precautions to insure accuracy. Yet it seems well to furnish answers to some of the first problems under each head,

because the learner needs more encouragement and assistance at the outset; but he should aim to become master of the subject before leaving it, and one of the tests of this is the ability to give an independent answer.

It will be observed, too, that no special provision is made for "mental" or "intellectual" exercises (so called) apart from the exercises of practical arithmetic. Perhaps no feature of arithmetical instruction is more popular than this, and I fully appreciate the reluctance which most will feel even to question its usefulness. But the conviction has been reached only with deliberation, and after long experience and observation, that much time has been wasted in so-called "arithmetical drill."

If it be admitted that the arithmetical student becomes accomplished when he has attained these three degrees of skill: 1°, when he is able to perform all the simple operations with facility and accuracy; 2°, when he becomes familiar with the "material" in the use of which numbers find their application; and 3°, when he becomes familiar with the usual applications to practical problems; then it is fair to ask why more than this should be exacted. It is intended to present in this treatise, in theory and practice, all that belongs to the three parts named.

Actual trial has shown the efficiency of the method here sketched, and there is no apparent reason why it should not continue to be efficient.

With these words of explanation my book is sent forth in the hope that it may fulfill a mission of usefulness.

SUGGESTIONS TO TEACHERS.

In Part I. it is assumed that the pupil is competent to understand the principles of simple numerical operations when clearly presented.

In general the learner should not attempt to advance beyond the limit of his ability to comprehend. It is better, in such case, to lay aside the subject for a while, and instead to engage in the study of something within the grasp of the intellectual powers. The first principles of Natural History furnish an inexhaustible amount of material suitable for such purpose. In this part of arithmetical study the pupil should commit to memory the multiplication table, but scarcely anything more should be memorized.

For the present avoid problems, and in selecting exercises do not choose those which are beyond the pupil's ability to master with reasonable effort, or which are on the extreme verge of this limit.

If an exercise cannot be readily worked, let the pupil erase the first result, and, if necessary, work it again several times. He will learn more in working one exercise half a dozen times in such case than by working half a dozen exercises, each one of which he is scarcely able to master.

Keep in mind that the *first object is to acquire facility and*

accuracy in adding, multiplying, subtracting, and dividing, and chiefly in the two first named, since subtraction is an inverse operation of addition, and division an inverse operation of multiplication.

Some familiarity in the use of multiples and factors is necessary as a preliminary to operations upon fractional numbers.

In Part II. the pupil should learn all the tables in common use—should finally memorize them—and should be able to make readily all reductions from given denominations to others required.

In Part III. greater demands are made upon the reasoning powers of the student, and if he is well skilled in the simple operations, and familiar with the reductions of compound denominations, he can work the more effectively.

First of all, the pupil should aim to understand each principle, if possible, without aid from the teacher; but if that becomes necessary, let it be as judicious as possible, and always by suggestion rather than by direct statement.

The pupil, having learned the principle and solved the problems relating to it, should be required habitually to give an explanation of the principle and give an analysis of the problems. By this means he not only insures a more thorough knowledge of the subject itself, but becomes skilled in the art of accurate expression.

PART I.

SIMPLE OPERATIONS.

COURSE IN ARITHMETIC.

PART I.

SIMPLE OPERATIONS.

CHAPTER I.

First Principles of Numbers.

EVERY one, no doubt, who begins the study of Arithmetic, can count one, two, three, or more, and knows the meaning of the words "how many."

Should the question be asked, "How many dollars have you?" the answer might be *one* dollar, or it might be *ten*, or *one-fourth*, or *six-and-a-half*, and the terms *one*, *ten*, *one-fourth*, and *six-and-a-half*, are called numbers.

Any term which denotes how many things are thought of is called a *number*. *One* denotes a single thing; *ten* denotes a collection of several things; *one-fourth* denotes a part of a thing; while *six-and-a-*

half denotes whole things and a part of a thing considered together.

There is no end to the series of numbers that may be named, but it is evident that numbers must denote whole things or parts of things, or the two considered together.

Numbers denoting whole things are called *whole numbers*, or sometimes *integers* or *integral numbers*, and numbers denoting parts of things are called *fractional numbers*. Numbers denoting whole things and parts together are called *mixed numbers*.

A single thing is often called a unit, and numbers are sometimes said to express units; but though numbers express units, numbers *are not units*, and the learner should carefully distinguish between numbers and the things numbered.

Numbers are useful in many ways and among all classes of people. Bankers count money, boys count marbles, and the Indians count the buffalo skins which they sell to traders, all by means of numbers.

Sometimes the reckoning is easy, but often so many different numbers are to be used in one reckoning that mistakes would occur if there were not convenient methods of writing and using the numbers. In the pages that follow, the usual methods of reckoning numbers are explained, and this book is called a treatise on Arithmetic, because it treats of numbers and the methods of using them.

REVIEW I.

a. **A number is** a term which denotes how many things are thought of.

b. **A number may denote** a whole thing or whole things, or a part or parts of things, or the two together.

c. **An integer, or integral number,** denotes only whole things.

d. **A fractional number** denotes a part or parts of things.

e. **A mixed number** denotes whole things and parts, taken together.

f. **A unit** is a single thing, and though numbers may express a unit or a collection of units, it is not correct to say that number is a unit or a collection of units.

g. **Arithmetic** treats of numbers and of the methods of using them.

CHAPTER II.

Numeration, or Naming Numbers.

There is no end to the series of possible numbers, and because a great many are required for use, it is the more important to provide convenient names for them.

Any *system* of *naming numbers* is called *numeration.* Among nearly all nations the systems of numeration have had one principle in common—that is, the forming of orders-of-numbers by tens. This uniformity has probably resulted from the number of fingers on the two hands, which doubtless furnished the most convenient means for counting for people in the primitive condition of society.

The principle of orders-of-numbers formed by tens will be easily recognized in the system of numeration in common use, which will now be explained.

The first ten whole numbers in the natural order are one, two, three, four, five, six, seven, eight, nine, ten. Then follow eleven, twelve, thirteen (or three and ten), fourteen (or four and ten), fifteen, and so on to twenty or two tens. Then twenty-one, twenty-two, twenty-three, and so on to thirty or three tens. Then thirty-one, thirty-two, thirty-three, and so on to forty or four tens. In a similar manner to fifty or five tens, and sixty or six tens, seventy or seven tens, eighty or eight

tens, ninety or nine tens, then to ten tens, which are called one hundred.

So far, then, the name of any number, if it is more than ten, indicates a number of tens together with a number less than ten, unless it happens to be an exact number of tens. Following one-hundred, the names of numbers are formed by uniting to one-hundred the names of each of the preceding numbers from one to ninety-nine successively, as one-hundred-one, one-hundred-two, one-hundred-three, etc., to one-hundred-ninety-nine, after which the next number is called two-hundred. From two-hundred to three-hundred the names are formed as before, by uniting successively to two-hundred the names of the numbers from one to ninety-nine. In a similar way to four-hundred, to five-hundred, six-hundred, seven-hundred, eight-hundred, nine-hundred, and then ten-hundred, which is called a thousand. From one-thousand to two-thousand the names of numbers are formed by uniting successively to one-thousand the names of each of the preceding numbers.

In a similar manner the names of numbers proceed to three-thousand, four-thousand, and so on to ten-thousand; then eleven-thousand, twelve-thousand, and so on to twenty-thousand, thirty-thousand, and so on to ten ten-thousand, or one-hundred-thousand. Then from one-hundred-thousand to ten-hundred-thousand, which is called a million.

From the foregoing it appears that the names of numbers are based upon orders of tens, and tens of tens, and higher orders, as follows:

A single thing is called a unit, or one thing.

Ten ones are called a ten.

Ten tens are called a hundred.

Ten hundreds are called a thousand.

Ten thousands are called ten-thousand.

Ten ten-thousands are called a hundred-thousand.

Ten hundred-thousands are a million.

Ones, tens, hundreds, thousands, etc., are called orders-of-numbers, and ten of any one of these orders are equivalent to one of the order next above.

It also appears that any whole number is expressed in words by naming successively the numbers of the different orders that compose the number, beginning first with the highest order. Thus we say, one-thousand-eight-hundred-seventy-six, remembering that the names of the second order (or order of tens) are usually contracted—for instance, instead of *seven tens* we say *seventy*, instead of *three tens* we say *thirty*, and so on; and further, that numbers between ten and twenty are expressed by a single word, as for instance, *eleven* instead of *ten-one*, *twelve* instead of *ten-two*, and so on.

The orders-of-numbers have already been named to millions, and beyond that the orders are ten-millions, hundred-millions, billions, ten-billions, hundred-billions, trillions, quadrillions, quintillions, sextillions, septillions, octillions, nonillions, decillions, and so on.

These names of orders-of-numbers are derived from the Latin numerals, and can be formed still further without limit.

The system of numeration just explained, which is

in common use in this country, is adopted from the French, and differs from the English in the names given to the orders of numbers above the ninth, as shown below:

	French System.	*English System.*
Ninth order.......	hundred-millions.	hundred-millions.
Tenth order.......	billions.	thousand-millions.
Eleventh order....	ten-billions.	ten-thousand-millions.
Twelfth order.....	hundred-billions.	hundred-thousand-millions.
Thirteenth order ..	trillions.	billions.
Fourteenth order..	ten-trillions.	ten-billions.
	etc.	etc.

So far only integrals or whole numbers have been considered, but now we will notice briefly the numeration of fractional numbers.

Fractional numbers are used to denote parts of things, and in order to represent a portion of any thing, the thing is supposed to be separated into some number of equal parts, and a sufficient number of these equal parts are taken to express the required portion.

Thus a thing may be separated into two equal parts called halves, or into three equal parts called thirds, or four equal parts called fourths, or into any other number of equal parts, and generally any portion of a thing may be expressed by taking some number of the equal parts into which the thing may be separated. The name given to one of these equal parts is usually formed by uniting *th* to the name of the number which expresses the entire number of equal parts. For example, one of four equal parts is called a *fourth*, one of six equal

parts is called a *sixth*, one of seven equal parts a *seventh*, and so on.

Among the exceptions to this rule may be noticed one of the two equal parts called a half, one of three equal parts called a third, one of five equal parts called a fifth. But higher numbers whose names end in *one*, or *two*, or *three*, follow the usual rule, as one of twenty-one equal parts called a twenty-oneth, or one of twenty-two equal parts called a twenty-twoth, or one of twenty-three equal parts called a twenty-threeth, and so on.

A fractional number is named by uniting the name of the number of equal parts used or considered to the name of one of the equal parts. Thus, three-fourths denotes three of the equal parts called fourths.

The number of equal parts into which a thing is supposed to be separated is called the denominator, and the number of equal parts used or considered is called the numerator. In the case of three-fourths, four is the denominator, and three is the numerator; or in the case of five-sevenths, seven is the denominator and five is the numerator.

REVIEW II.

a. **Numeration** is a system of naming numbers.

b. **One common principle** has been used in the systems of nearly all nations—that of forming orders-of-numbers by tens.

c. **The orders-of-numbers in common use** are as follows:

First order......	units, or single things.
Second order....	tens.
Third order.....	hundreds.
Fourth order....	thousands.
Fifth order	ten-thousands.
Sixth order	hundred-thousands.
Seventh order...	millions.
Eighth order....	ten-millions.
Ninth order.....	hundred-millions.
Tenth order.....	billions.

And so on, in which ten of any one order are equivalent to one of the next order higher.

d. **Any whole number is expressed** by naming successively the numbers of different orders that form parts of that number, naming first the number of highest order.

e. **The names of the orders-of-numbers** above the order of hundred-billions are derived from the Latin numerals, and may be formed to an indefinite extent.

f. **In the French system of numeration,** a thousand millions are called a billion, a thousand billions are called a trillion, a thousand trillions are called a quadrillion, and so on; while in the English system, a million of millions form a billion, a million of billions form a trillion, a million of trillions form a quadrillion, and so on.

g. **A fractional number is expressed** as some number of the equal parts into which a thing is supposed to be separated.

h. **The name of one of these equal parts** is usually formed by annexing *th* to the name of the number which expresses all the equal parts, and a fractional number is named by uniting the name of the number of equal parts to the name of one of the equal parts.

CHAPTER III.

Notation, or Writing Numbers.

IN the preceding chapter the names of numbers used are written in words, as other names are written, but for the purposes of reckoning, more convenient methods of expressing numbers have long been in use.

Any method of expressing numbers by signs, or by marks called figures, is notation.

The Greeks used letters of their alphabet to denote numbers, as likewise did the Romans, and the Roman notation is still in use, but chiefly for numbering the chapters and sections of books, and public documents, and for similar purposes.

In the Roman method seven capital letters are used, viz., I, V, X, L, C, D, M, representing, respectively, one, five, ten, fifty, one hundred, five hundred, one thousand.

By repeating them, or combining them in various ways, any whole number can be expressed, according to the manner shown in the following table:

I.	denotes	one.	VII.	denotes	seven.
II.	"	two.	VIII.	"	eight.
III.	"	three.	IX.	"	nine.
IV.	"	four.	X.	"	ten.
V.	"	five.	XI.	"	eleven.
VI.	"	six.	XII.	"	twelve.

XIII.	denotes	thirteen.
XIV.	"	fourteen.
XV.	"	fifteen.
XVI.	"	sixteen.
XVII.	"	seventeen.
XVIII.	"	eighteen.
XIX.	"	nineteen.
XX.	"	twenty.
XXI.	"	twenty-one.
XXII.	"	twenty-two.
XXX.	"	thirty.
XL.	"	forty.
L.	"	fifty.
LX.	"	sixty.
LXX.	"	seventy.
LXXX.	"	eighty.
XC.	"	ninety.
C.	"	one hundred.
CI.	denotes	one hundred-one
CX.	"	one hundred-ten
CC.	"	two hundred.
CCC.	"	three hundred.
CCCC.	"	four hundred.
D.	"	five hundred.
DC.	"	six hundred.
DCC.	"	seven hundred.
DCCC.	"	eight hundred.
DCCCC.	"	nine hundred.
M.	"	one thousand.
MM.	"	two thousand.
MDCCCLXXVI.	denotes	one thousand eight hundred seventy-six.
$\overline{\text{V}}$.	denotes	five thousand.
$\overline{\text{X}}$.	"	ten thousand.

From this table it appears that repeating a letter, in effect adds the value represented by the letter; as, for instance, X. denotes ten, and XX. denotes twenty, or twice ten.

A letter placed before another representing larger value diminishes that value, but when placed after it increases that value, as IX. denotes nine, and XI. denotes eleven.

A bar placed above a letter indicates thousands.

These explanations will, however, be better understood after some practice in writing numbers according to this method, and the following list is for that purpose:

1. Forty-five.
2. One hundred twenty-seven.
3. Seventy-three.

4. Six hundred eighteen.
5. Eighteen hundred ninety-two.
6. Nineteen hundred one.
7. Thirty-nine.
8. Five thousand five hundred fifty-five.
9. Two thousand seventeen.
10. Six hundred thirty-seven.

An exercise in reading numbers expressed by the Roman notation:

1. LXXVII.
2. MX.
3. MC.
4. LXIV.
5. XCIX.
6. CXIX.
7. CLIV.
8. CXLIX.
9. DCLX.
10. MDCCCLXXVII.

But the notation in common use, and which is more convenient for most purposes, is called the Arabic. It is so called because modern Europeans, from whom we have derived it, obtained their first knowledge of it from the Arabs, though it is now known that the Arabs learned it from the Hindoos, who have used it for several thousand years.

In this system ten marks or characters, called figures, are used, as follows:

0,	1,	2,	3,	4,
naught, or cipher,	one,	two,	three,	four,

5,	6,	7,	8,	9.
five,	six,	seven,	eight,	nine.

By means of these figures, either singly or combined in different ways, any number that can be named may be easily expressed.

Any whole number (we have seen) is expressed in words by naming successively the numbers of the different orders that compose it.

The number of any one of these orders is less than ten, and can, therefore, be expressed by some one of the ten Arabic figures.

Any whole number whatever is expressed in the Arabic notation by placing one after another the figures which express the respective orders of numbers, placing the figure of highest order at the left, and so on to the figure of lowest, or units order, which is placed at the right.

Thus, thirty-six, or three tens and six units, may be expressed 36, and three hundred twenty-four may be expressed 324, the figure 3 in this case denoting *hundreds*, the figure 2 denoting *tens*, and the figure 4 denoting *ones*, or *units*.

In all cases the place of the first figure at the right denotes units, the second place from the right denotes tens, the third place denotes the order of hundreds, and so on.

It is important to remember, in case any of the intermediate orders of numbers do not appear in the given number, a cipher or naught must appear in the figures to fill the place belonging to the order.

For instance, in the number three hundred two, the order of tens does not appear; therefore, in the figures expressing this number, the second place belonging to the order of tens must be filled by a cipher, thus, 302.

So also four thousand four, is expressed, 4004; the places of tens and hundreds both being occupied by ciphers.

A careful study of the following series of numbers, expressed both in words and figures, will be useful to the learner.

10—ten.	24—twenty-four.
11—eleven.	25—twenty-five.
12—twelve.	26—twenty-six.
13—thirteen.	27—twenty-seven.
14—fourteen.	28—twenty-eight.
15—fifteen.	29—twenty-nine.
16—sixteen.	30—thirty.
17—seventeen.	31—thirty-one.
18—eighteen.	40—forty.
19—nineteen.	50—fifty.
20—twenty.	100—one hundred.
21—twenty-one.	200—two hundred.
22—twenty-two.	1000—one thousand.
23—twenty-three.	124—one hundred twenty-four.

1876—one thousand eight hundred seventy-six.

2001—two thousand one.

It should be noticed that the number ten is expressed by two figures, though containing only one order of number, because that order is the second or order of tens, and the place of the first order is occupied by a cipher. From ten to twenty, each number contains two orders of numbers, though expressed by a single word, as *eleven*, which is composed of one *ten* and one, and expressed in figures, 11.

For convenience of reading numbers expressed by many figures, it is usual to mark the figures by commas into groups or periods of threes, from the right.

Thus, having the figures 42345724680, they may be formed in groups, as follows:

billions,	millions,	thousands,	units.
42,	345,	724,	680

and read, forty-two billions, three hundred forty-five millions, seven hundred twenty-four thousands, six hundred eighty.

The learner will find it useful to practice writing numbers according to the Arabic notation, and also to read them.

Exercises of both kinds are given below.

Until some skill is acquired in the art of reading and writing numbers, it will be found of advantage to write the names of the orders of numbers, above the groups of threes, as in the example last given.

After some familiarity has been gained with the exercise, this may be omitted.

Notation of Fractional Numbers.

We have learned that a fractional number is expressed in words, as some number of the equal parts of a thing, the entire number of equal parts being called the denominator, and the number of the equal parts used or considered being named the numerator.

A fractional number is expressed in the Arabic nota-

tion by placing the figures of the numerator *above*, and the figures of the denominator *below* a horizontal line. Accordingly the fractional number three-fourths is expressed by $\frac{3}{4}$; nineteen-twentieths by $\frac{19}{20}$, and so on.

It should be noted that the figures used in the Arabic notation, or the letters of the Roman notation, or any characters used to take the place of words, are often called symbols. Further on, we shall meet with symbols used to indicate operations upon numbers.

EXERCISES.

EXPRESS IN FIGURES.

(1). Fourteen thousand four hundred fourteen.

(2). Twenty millions two hundred thousand two.

(3). Seventeen trillions seven millions seven hundred seven.

(4). Nine hundred ninety-nine thousands ninety-nine.

(5). Fourteen millions fourteen thousands fourteen.

(6). One hundred fourteen millions three hundred ninety-seven.

To read numbers expressed in figures.

(1).	72,091.
(2).	1,414.
(3).	303,030,303.
(4).	10,101,010,101,010.
(5).	1,112,223,334,445,556.
(6).	998,877,665,544,332,211.
(7).	100,200,300,400,500,600.
(8).	10,002,000,300,040,005,000.
(9).	111,111,111,111,111,111,111.
(10).	123,456,789,098,765,432,101.

REVIEW III.

a. **Notation** is a system of expressing numbers by characters or figures.

b. **Two systems of notation** are in common use—the Roman and the Arabic.

c. **In the Roman notation** seven capital letters are used, viz.: I, V, X, L, C, D, M, expressing respectively the numbers one, five, ten, fifty, one hundred, five hundred, one thousand.

d. **These letters are combined according to three principles.** 1st. Repeating a letter repeats its value. 2d. Placing one letter before another of larger value diminishes that value, but placing it after one of larger value increases it. 3d. A line or bar placed above a letter signifies thousands.

e. **The Roman notation is used** chiefly for numbering chapters and sections of books and public documents.

f. **The Arabic notation is so called** because derived by modern Europeans from the

Arabs, though it is now known the Arabs learned it from the Hindoos.

g. **In the Arabic notation** ten marks or characters, called figures, are used, viz.: 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, signifying respectively naught, one, two, three, four, five, six, seven, eight, nine.

h. **The principles of the Arabic notation are:** 1st. Each one of the different orders of numbers which compose any whole number is expressed by a single figure. 2d. The figures are placed one after another, the figures of highest order at the left, and so on to the figure of the lowest, which is placed at the right. 3d. The absence of any order of number lower than the highest must be indicated by a cipher in the place representing that order.

i. **For convenience of reading** it is customary to separate the figures by commas into groups or **periods of three each.**

j. **To express a fractional number** in the Arabic notation, place the figures of the numer-

ator above a line, and the figures of the denominator below the same line.

k. Figures or other characters used in place of words to express numbers are called **symbols**. There are also symbols of **operations upon numbers**.

CHAPTER IV.

Addition, or Counting Numbers Together.

The first and most simple operation upon numbers is to count two or more numbers together. This process is called *addition*. Thus, suppose there are three apples in one dish and two in another. It is easy to count the two groups together, and five is the result. The number obtained by addition is called the sum.

In case there are three numbers of things to be added, any two may be first added, and the third may then be added to the sum of the first two.

It often happens that the kind of things the numbers of which are added is not mentioned, but in every case some kind of thing is understood. If it be said, "*Two and three are five,*" the statement means simply that "*two things and three things counted together are five things,*" and there is no arithmetical addition of numbers *except as numbers of things.*

The process of addition is then purely mental. The placing of things side by side does not constitute the process, but the counting them together.

But it is found, though the process is carried on mentally, great assistance is often obtained by writing the figures of the numbers to be added, and their sum. This is the case if the numbers are large, or if there are many small numbers to be added together. In

any case the student should be so familiar with the sums of small numbers as to remember the sum of any two less than ten.

It is found most convenient to write the numbers to be added in a vertical line or column. If it be required to add together 2, 4, and 7, they may be written as follows:

$$\begin{array}{r} 2 \\ 4 \\ 7 \\ \hline 13 \end{array}$$

And drawing a line underneath, the sum 13 is written below it. This line is drawn in order to distinguish readily the numbers to be added from their sum, and thus avoid confusion.

Suppose the numbers to be added are greater than ten, as 24 and 48. Again writing these so that the figures of the same order fall in the same column, we have

$$\begin{array}{r} 24 \\ 48 \\ \hline \end{array}$$

Each of the given numbers is composed of a number of tens and a number of units, and it is found more convenient to add first the units and then the tens. The sum of 8 and 4 is 12; that is 1 ten and 2 units; and the sum of 4 tens and 2 tens is 6 tens. So the operation may be expressed as follows:

$$\begin{array}{rl} & 24 \\ & 48 \\ \hline & 12 = \text{sum of units.} \\ & 60 = \text{sum of tens.} \\ \hline \text{or} & 72 = \text{total sum.} \end{array}$$

It is obvious that the unit figure 2 of the sum of units might first be written alone, and the 1 ten reserved without writing it, and added at once to the sum of the tens. In this way the operation would be expressed as follows:

$$\begin{array}{r} 24 \\ \underline{48} \\ 72 \end{array}$$

and may be described by saying, "I first add the units 8 and 4, whose sum is 12, or 1 ten and 2 units. I write the 2 units in the column of units, and reserve 1 ten to be added in with the sum of tens. That is, 1 ten, 4 tens, and 2 tens added together are 7 tens, which is now written in the column of tens, and the entire result is 72."

Let it be required to add 437, 684, and 327.

Proceeding as before, writing one under another, and adding the numbers of separate orders, beginning with units, we have—

$$\begin{array}{r} 436 \\ 648 \\ \underline{327} \\ 1411 \end{array}$$

The sum of the units 7, 8, and 6 is 21, or 2 tens and 1 unit. The unit 1 is written in the column of units, and the 2 tens reserved to be added in the column of tens. The sum of the tens 2, 4, and 3 is 9 tens, with which, adding 2 tens reserved from the sum of the units, the result is 11 tens, or 1 hundred and 1 ten. Writing 1 ten in the column of tens, the 1 hundred is reserved to be added in with the hundreds. The sum

of the hundreds 4, 6, and 3, with 1 hundred reserved, 14 hundreds, or 1 thousand 4 hundred, which is written in the result as the given numbers are now all added, and the entire result is 1411.

Of course the reserved number may be added with the first instead of the last of the numbers of any column.

In a similar way any other numbers may be added.

From the foregoing it is easy to deduce the rule for adding whole numbers, as follows:

Write the numbers to be added, one under another, so that figures of the same order shall fall in the same column. Then draw a line underneath the whole, and add first all the units, then all the tens, and all the hundreds, and so on.

If any partial sum is less than ten of the order added, write it underneath in its proper order. If any partial sum is ten or more than ten of the order added, reserve the number of *tens* to be counted as *ones* of the next higher order, and write the remaining portion which is less than ten in its proper order in the result. The ten or tens reserved from any partial sum will then be counted with the numbers of the next higher order. So proceed till all the orders of numbers are added, writing the last partial sum in full.

It is found by experience that mistakes often occur in adding series of numbers, even after considerable practice, and it becomes desirable to have some method for finding and correcting such mistakes, or as it is sometimes said, for *verifying* the work.

A method in frequent use is to add the numbers expressed in each column in the inverse order. That

is, if the addition in the first instance is from the foot of the column upwards, the second is from the top downwards. If the two results agree it is highly probable no error has occurred; but if they differ one or the other must be wrong, and the work should be carefully examined and the two results made to coincide.

To indicate that two numbers are to be added a sign or symbol is used, formed by a vertical line crossing a horizontal one, thus, $+$.

This sign, named *plus*, is placed between the symbols of the numbers to be added. Thus $14+23$ is read *fourteen plus twenty-three*, and means *fourteen added to twenty-three.*

Two equal horizontal lines, one above the other, thus, $=$ form the *symbol* of *equality*, and are read "*equal to*," or "equals." So $4+3=7$, is read, four plus three equals seven.

Such signs or symbols are more easily written than the words for which they stand, and occupy less space, and are therefore more convenient.

Skill in adding numbers can only be acquired by practice, after the foregoing explanations are understood, but some hints may be useful to the learner. Suppose now we have a series of numbers to be added, arranged in order, with a line underneath, as follows:

$$\begin{array}{r} 2358 \\ 9426 \\ 837 \\ 2332 \\ \hline 14953 \end{array}$$

Perhaps the learner, beginning at the right hand column, would say, "2 and 7 are 9;" "9 and 6 are 15;" "15 and 8 are 23," which is the sum of the numbers expressed in the first column. But it would be better, pointing successively to the figures 2, 7, 6, 8, to say "2, 9, 15, 23," performing the partial additions mentally, and pronouncing only the partial sums.

The figure 3 is written in the unit's place in the result, and the two tens counted in with the column of tens. Adding 2 tens to the 3 tens, and proceeding as before, pronounce the partial sums, 5, 8, 10, 15, and 15 tens is the sum of all the tens. Reserving ten tens, 5 is written in the result, and 1 hundred (as 10 tens) is counted with the hundreds, pronouncing the partial sums, 4, 12, 16, 19. Here again write 9 hundreds and count 10 hundreds as 1 thousand, with the thousands. Finally the sum of the thousands is in a similar way found to be 14, which, written in the result, completes the process.

Some computers like to write the numbers reserved from the partial sums above the columns with which they are to be counted. Thus in the last example, drawing a line above the column of figures and writing the several numbers, we have

$$\begin{array}{r} 112 \\ \hline 2358 \\ 9426 \\ 837 \\ 2332 \\ \hline 14953 \end{array}$$

Whether the reserved numbers are written or not is

merely a question of convenience, which each one will decide according to his judgment.

Again, computers find that with practice it becomes easy to add the numbers expressed in two columns at one operation, and thus gain in rapidity. For instance, using the same example again, we may count the two columns at once, and say, "32, 69, 95, 153," and writing 53, count 1 with the next two columns, where the partial sums are 24, 32, 126, and finally 149, which is written in its place, making the same result as before obtained.

So the numbers of three and even four columns may be added at once, but in ordinary cases one column at a time is enough, and the learner should first acquire skill in this before attempting more.

EXERCISES IN ADDITION.

Add the numbers as indicated below and verify the results.

(1).	(2).	(3).	(4).	(5).
26	86	56	812	283
53	42	83	463	967
14	18	24	702	72
86	57	58	595	428

(6).	(7).	(8).	(9).
183	189	48972	120011
675	456	2708	1000
782	287	16021	10040
187	704	42333	299707
908	455	7777	405060
367	372	9191918	228000

(10). 1111111
2222222
3333333
4444444
5678901
3456789
1234567

(11). Add together 141414, 29764832, 56567878, 1235, and 120000

REVIEW IV.

a. **Addition** is the process of counting two or more numbers of things together, to form one number of things.

b. **The number obtained by addition,** is called the sum or amount.

c. **For the sake of convenience,** numbers to be added are written so that figures expressing the same order shall fall in one column or vertical line.

d. **In adding numbers,** first add all the units, then all the tens and higher orders in succession, being careful to reserve from any partial sum of the numbers of any order the

tens of that order, to add as ones with numbers of the next higher order.

e. **To verify** any operation is to establish its accuracy.

f. **To verify any process of addition,** add the numbers of the columns in an inverse order, when the second result should be the same as the first.

g. **The sign used to indicate addition** is a cross formed by a vertical line intersecting a horizontal one, thus, +. It is placed between the symbols of the numbers to be added, and is called *plus.*

h. **The sign of equality is formed** by two equal horizontal lines, one above the other, thus, =. When used it is placed between the symbols of equal numbers, and is read, "equal to" or "equals."

CHAPTER V.

Subtraction, or Finding the Difference between Two Numbers.

It is often necessary to find the difference between two numbers of things; that is, to find how many must be added to the smaller of two numbers, so that the sum shall be equal to the larger. For example, to find the difference between 7 and 4 is to find what number must be added to 4 that the sum shall be 7.

This process is called subtraction. The larger number is called the *minuend*, the smaller is called the *subtrahend*, and the result obtained is called the *difference.* The difference is often called the *remainder*, because the subtrahend is sometimes said to be *taken from* the minuend, in which case the difference would be a remainder.

The word subtract means primarily to *withdraw from*, and the name subtraction was given to this process of finding the difference between two numbers, because it was thought to describe it exactly, but it fails to do so. Yet in every case the process of subtraction is a process of finding the difference, though the name subtraction is retained in use. The nature of the process indicated by the name should, however, be clearly understood.

It is obvious that in finding the difference between two numbers of things the things must be of the same kind. For example, the difference between 7 sheep and 4 sheep is 3 sheep; but the difference between 7 sheep and 4 dogs is not arithmetical, for nothing can be added to the one to make a sum equal to the other.

One number is said to be subtracted from another when the difference is found; and the smaller number is subtracted from the larger.

So we may say, "To find the difference between 7 and 4;" or we may say, "To subtract 4 from 7," and the two forms of expression are equivalent.

To indicate subtraction a short horizontal line, or dash, thus, —, and named *minus*, is placed between the symbols of the numbers, the larger number, or minuend, being written first. So, 7—4 is read "7 minus 4," or "7 less 4," and means that the difference between 7 and 4 is required.

The process of subtraction is easy when both minuend and subtrahend are small numbers, for in that case the number which must be added to the smaller to make a sum equal to the larger is readily found. For example, to find the difference between 7 and 4, one remembers that "4 and 1 are 5," and "4 and 2 are 6," "4 and 3 are 7," and therefore 3 is the number sought.

It is essentially such a process the mind is accustomed to go through until the difference between small numbers becomes familiar.

The method of subtracting in the case of large numbers is made to depend directly upon the difference of small numbers.

Suppose the minuend to be 548, the subtrahend 125. The number sought must be such that, added to 125, the sum shall be 548.

Let us write the figures of the subtrahend, for the sake of convenience, so that figures of the same order shall fall in the same column, thus:

$$\begin{array}{r} 548 \\ 125 \\ \hline 423 \end{array}$$

and draw a line underneath. It is at once seen that 3 units added to 5 units gives the result 8 units. That 2 tens added to the 2 tens of the subtrahend will make 4 tens of the minuend, and that 4 hundred added to the 1 hundred of the subtrahend will make the 5 hundred of the minuend. It is therefore clear that 423 added to the subtrahend will give an amount equal to the minuend, and is therefore the difference sought.

In this case the process of subtraction may be described as follows:—Write the figures of the subtrahend under those of the minuend, so that the figures of the same order fall in the same column, then subtract the numbers expressed in each respective column, separately, writing each partial difference directly underneath. The result thus obtained is the difference sought.

But an apparent difficulty occurs when it happens that the number expressed by any figure of the subtrahend is greater than that expressed by the figure of the same order in the minuend.

Suppose, for example, it is required to subtract 463

from 846. Writing the figures of the subtrahend under those of the minuend, we have

$$\begin{array}{r} 846 \\ \underline{463} \end{array}$$

Now there is no difficulty in subtracting 3 units from 6 units, but there is a difficulty in subtracting 6 tens from 4 tens. But the difficulty would be obviated if in any way the number of tens in the minuend should be caused to be more than the number of tens in the subtrahend. To do this 1 hundred is taken from the 8 hundreds of the minuend, and, regarding it as 10 tens, is added to the 4 tens, making 14 tens, and this is more than the number of tens in the subtrahend.

To show this more clearly the minuend may be expressed as follows:

Hundreds.	Tens.	Units.
8	4	6

But making the exchange above explained, taking 1 hundred from the number of hundreds and adding 10 tens to the number of tens, we may write again,

	Hundreds.	Tens.	Units.
Minuend	7	14	6
Subtrahend	4	6	3
Difference	3	8	3

The subtraction is now easily accomplished, and the difference is found to be 383.

When the principle is once clearly understood it will not be necessary to write out the process in the manner

just described, but it may be described as below, the numbers being expressed in the usual manner:

$$\begin{array}{r} 846 \\ \underline{463} \\ 383 \end{array}$$

Subtracting 3 units from 6 units, the difference is 3 units, which is written underneath in the same column. 6 tens cannot be subtracted from 4 tens, but taking 1 hundred from the 8 hundred and adding it as 10 tens to the 4 tens, gives 14 tens, from which 6 tens can now be subtracted, and the difference, 8 tens, is written below in the column of tens. Finally, subtracting 4 hundreds from the remaining 7 hundreds, the difference is 3 hundred, which is written underneath, and the entire result is 383.

It may happen that there is nothing in the next order of number in the minuend from which to take.

Suppose it were required to subtract 125 from 4004. The minuend may be written,

Thousands.	Hundreds.	Tens.	Units.
4	0	0	4

Or, again, it may be written,

	Thousands.	Hundreds.	Tens.	Units.
Minuend......	3	9	9	14
Subtrahend....		1	2	5
Difference.....	3	8	7	9

In this case, as 5 units cannot be subtracted from 4 units, and as there are no tens nor hundreds from which to take, it is necessary to take from the next

higher order 1 thousand, or 10 hundreds. From 10 hundreds take 1 hundred, or 10 tens, leaving 9 hundreds, and from the 10 tens take 1 ten, leaving 9 tens, and adding to the 4 units makes 14 units. The difference is now easily obtained. It is found in practice to be easier to add 10 to any order of the minuend, and to add as compensation 1 to the next higher order of the subtrahend.

In the last example, for instance, the subtraction may be performed as follows:

$$\begin{array}{r} 4004 \\ 125 \\ \hline 3879 \end{array}$$

Since 5 units cannot be subtracted from 4 units, add 10 to the 4 units, making 14 units, when the difference, 9, is easily found. But having added 10 units to the minuend we may add 1 ten to the subtrahend, as compensation, making 3 tens in the subtrahend instead of 2 tens. As there are no tens in the minuend, add 10 tens, and subtracting 3 tens, the difference, 7 tens, is written in its place. To balance 10 tens which were added to the minuend, add 1 hundred to the subtrahend, making altogether 2 hundreds. Again adding to the minuend 10 hundreds, and subtracting 2 hundreds, the difference, 8 hundreds, is also written down. Finally, adding 1 thousand to the subtrahend to balance the 10 hundreds added to the minuend, and subtracting it from 4 thousands, the difference, 3 thousands, is written down, completing the result, which is the same as previously obtained.

When the learner becomes familiar with this process, he will understand it if described more briefly, as follows:

5 cannot be subtracted from 4, but subtracting 5 from 14 the result is 9. Carrying 1 to 2 makes 3, and 3 from 10 gives 7. Carrying 1 to 1 makes 2, and 2 from 10 gives 8. Carrying 1 to 0 is 1, and 1 from 4 gives 3, and the result is 3879.

Errors may occur in subtraction as well as in addition, and some means of verifying the work is necessary. The usual method is to add the difference to the subtrahend, and their sum should be equal to the minuend.

Applying this test to the preceding example the result is as follows:

Subtrahend	125
Difference	3879
Minuend	4004

EXERCISES IN SUBTRACTION.

(1).	From	2412	subtract	1832....	*Ans.*	580
(2).	"	2884	"	1885....	"	999
(3).	"	45566	"	16667....	"	28899
(4).	"	63322	"	18878....	"	44444
(5).	"	76543	"	22222....	"	54321
(6).	"	10010	"	7788....	"	.....
(7).	"	11202	"	3434....	"	.....
(8).	"	10001	"	7007....	"	.....
(9).	"	98009	"	11111....	"	.....
(10).	"	1666666	"	779989....	"	.....

REVIEW V

a. **Subtraction** is the process of finding the difference between two numbers, or of finding what must be added to one of two numbers that the sum shall equal the other.

b. **The minuend** is the larger of the two numbers whose difference is sought.

c. **The subtrahend** is the smaller of the two numbers whose difference is sought.

d. **The difference** is the number sought, which, added to the subtrahend, shall make a sum equal to the minuend.

e. **The difference is often called the remainder** because the subtrahend is sometimes said to be *taken from* the minuend.

f. **The literal meaning of the word subtract** is to "withdraw from," and the name "subtraction" is retained in arithmetic to describe the process of finding the difference between two numbers, even though this difference be not found by any process of "withdrawing from."

g. **The minuend and subtrahend** must be numbers of the same kind of thing.

h. **The sign of subtraction** is a short horizontal line placed between the figures of the minuend and subtrahend, and is named *minus*.

i. **The method of subtraction** in the case of large numbers is made to depend directly upon that of small numbers whose differences are less than ten.

j. **For convenience in subtracting,** the figures of the subtrahend are written under those of the minuend, so that those of the same order shall fall in the same column. Beginning with the order of units, subtract the number of each order in the subtrahend from the number of the same order in the minuend.

If it happen that the number in any order in the subtrahend is smaller than the corresponding number in the minuend, add 10 to that number in the minuend before subtracting, and then add 1 to the number of the next higher order in the subtrahend.

Each partial difference is written underneath and in its proper order. The partial differences will then together express the total difference sought.

k. **To verify any process of subtraction** add the subtrahend and difference together, and if the work is correct their sum will be equal to the minuend.

CHAPTER VI.

Multiplication, or Counting a Number of Things a Number of Times Together.

IN the process of addition two or more numbers of things are counted together to form a single number, but it is sometimes necessary to count the same number several times together also to form a single number. This process is called *multiplication.* The number of things to be counted is called the *multiplicand.* The number denoting how many times the multiplicand is to be counted is called the *multiplier;* and the result obtained by multiplication is called the *product.* It is important to notice the essential difference in the processes of addition and multiplication.

In the first there is an operation upon *two or more numbers of things;* in the second there is an operation upon only *one number of things*, which may be repeated a *number of times.*

The multiplier is usually more than one, but it may be exactly one, or even less than one. Thus, 3 times 4 are 12. In this case 4 is the multiplicand, 3 the multiplier, and 12 is the product.

Again, once (or 1 time) 4 is 4. Here 4 is the multiplicand, 1 is the multiplier, and the product, 4, is the same as the multiplicand.

Suppose, now, the multiplier is less than one, say one-half, the multiplicand being 4 as before, and we say, "One-half of 4 is 2."

So it appears that counting 4 once (multiplying by 1), the product is the same—that is, 4; or counting 4 half a time (multiplying half a time), the product is 2.

As a practical illustration, suppose a man is using a rod 4 feet in length to measure the lengths of various pieces of timber.

He applies this rod 3 times successively along one of the pieces in the usual manner of measuring length, and so finds this piece to be 12 feet in length, because "3 times 4 are 12."

To another piece he applies the measure once only, and finds the piece to be 4 feet in length.

Again, measuring a shorter piece, he applies only half the measure, and finds this piece to be 2 feet in length, because one-half of (or $\frac{1}{2}$ time) 4 is 2.

Thus it appears that according as the multiplier is more than one, is one, or is less than one, the product is more than the multiplicand, is equal to the multiplicand, or is actually less than the multiplicand.

The product, then, is not always greater than the multiplicand.

The multiplicand and multiplier are often called the *factors*, or either one is called a *factor*, of the product.

In general a factor of a number is any one of the integral numbers which, multiplied together, will produce that number.

Usage sanctions the form of expression to "multiply by a number." The form, "Multiply a number of

times," would better express the fact. Thus to "multiply by 6" really means to "multiply 6 times." Both forms of expression will be found in what follows.

Before going further the learner should become familiar, if not already so, with the products of small numbers.

These products are shown in the table below, which is called a "Multiplication Table."

The form here given is known as the table of Pythagoras, so named from one of the ancient Greek philosophers, who is supposed to have devised it.

The series of whole numbers, from 1 to 12 inclusive, are written in the left hand column. In the next column are placed the products of each of these numbers multiplied 2 times, each product being written opposite the corresponding multiplicand.

In the third column the products obtained by multiplying 3 times are arranged in a similar way.

The products obtained by multiplying 4, 5, and 6 times, and so on respectively, are placed in the 4th, 5th, 6th, and following columns.

For example, the product of 8 times 5, which is 40, is found in the 8th column, and in the same horizontal line with the multiplier 5.

MULTIPLICATION TABLE.

1	2	3	4	5	6	7	8	9	10	11	12
2	4	6	8	10	12	14	16	18	20	22	24
3	6	9	12	15	18	21	24	27	30	33	36
4	8	12	16	20	24	28	32	36	40	44	48
5	10	15	20	25	30	35	40	45	50	55	60
6	12	18	24	30	36	42	48	54	60	66	72
7	14	21	28	35	42	49	56	63	70	77	84
8	16	24	32	40	48	56	64	72	80	88	96
9	18	27	36	45	54	63	72	81	90	99	108
10	20	30	40	50	60	70	80	90	100	110	120
11	22	33	44	55	66	77	88	99	110	121	132
12	24	36	48	60	72	84	96	108	120	132	144

Several Principles,

simple and obvious, yet important in many processes of multiplication, may be stated as follows:

First Principle.

In counting any given multiplicand any number of times (that is, in multiplying), each of the parts which make up that multiplicand is counted the same number of times. It follows, then, that the same result will be obtained if any multiplicand be separated into two parts, and these parts be multiplied separately, and their products added together, as obtained by the direct multiplication of the whole multiplicand.

For example, 4 times 8 are 32. But we may consider the multiplicand 8 as formed of two parts, 5 and

3; and counting these parts (multiplying) each 4 times, we have 4 times 5 are 20, and 4 times 3 are 12, and the sum of 12 and 20 is 32, as before.

That is, in 4 times 8, each unit of 8 is counted 4 times, and in 4 times 5, added to 4 times 3, each unit of 5 and each unit of 3 is also counted 4 times; in other words, the same number of units is counted the same number of times in either case.

The principle, then, may be stated as follows: *The product of the sum of any two numbers multiplied any number of times, is equal to the sum of the products of the separate numbers multiplied the same number of times.*

Second Principle.

It is also obvious, that if any multiplicand be multiplied as many times as the sum of two numbers, the product will be equal to the sum of the products of the multiplicand, multiplied the respective numbers of times.

Thus, 5 times 8 are 40. But separating the multiplier 5 into two parts, 2 and 3, and multiplying separately, we have 2 times 8 are 16, and 3 times 8 are 24, and $24+16=40$, as before. The same reasoning applies here as in the preceding case.

Third Principle.

Again, if the difference of two numbers be multiplied any number of times, the product will be equal to the difference of the respective products of the same numbers multiplied the same number of times.

Thus, if the difference between 8 and 5, which is 3, be multiplied 4 times, the product is 12. But the product of 4 times 8 is 32, and the product of 4 times 5 is 20, and the difference of the two products, 32 and 20, is 12, the same as 4 times 3, according to the principle just stated.

For it is evident in counting 8 once, and in counting 5 once, the difference is 3, and it is also evident that in four times counting them, the entire difference would be 4 times 3. The same reasoning would apply in case of any other numbers.

Fourth Principle.

By similar reasoning, it follows, if any numoer be multiplied as many times as the difference of two multipliers, the product will be equal to the difference of the products of the same numbers multiplied the respective numbers of times.

Thus, 5 times 8 are 40, and 3 times 8 are 24, and the difference of the two products is 16.

But the difference of the two multipliers, 5 and 3, is 2, and 2 times 8 are 16, according to the principle stated.

Fifth Principle.

The product is the same, whichever of two numbers is taken as the multiplicand, the other being taken as multiplier.

Thus, 3 times 4 is the same as 4 times 3; or 5 times 8 is the same as 8 times 5.

For if each unit of the multiplicand be counted

once for each unit of the multiplier, then each unit of the multiplier must be counted once for each unit of the multiplicand, which is what would take place if multiplicand and multiplier interchanged places.

For an illustration, take the numbers 3 and 4, and represent the separate units of 3 by the letters A, B, C, and the separate units of 4 by a, b, c, d.

Now, if each of the large letters be combined with each of the small letters once only, the resulting number will obviously be the same as though each small letter be combined once only with each large letter.

In the first case the result is—

Aa,	Ab,	Ac,	Ad,
Ba,	Bb,	Bc,	Bd,
Ca,	Cb,	Cc,	Cd.

In the second case—

Aa,	Ba,	Ca,
Ab,	Bb,	Cb,
Ac,	Bc,	Cc,
Ad,	Bd,	Cd,

and the number is evidently the same in each case.

Sixth Principle.

To multiply as many times as the product of any two factors gives the same result as to multiply first into one factor and then that product into the other factor.

Thus, to multiply 4, 6 times, amounts to the same as to multiply 4, 2 times, producing 8, and then multiply this product 8, 3 times, giving the result 24.

In this case the factors of 6 are 2 and 3, and mul-

tiplying 2 times, each unit of the multiplicand is counted 2 times, forming 2 units of that product. Now, multiplying this product 3 times, each of these sets of 2 units in the first product will form 3 times 2 units in the second product; that is, for each unit in the first multiplicand there will be 6 units in the final product. But this is what takes place in multiplying at once 6 times.

As the same reasoning would apply in the case of any other factors, the principle may be considered established.

Further, the same principle is obviously true for three or more factors, since the product of any two factors may be treated as a single factor.

So the principle may be stated—

To use the product of two or more factors as a multiplier, gives the same result as to use the factors successively.

To indicate multiplication, a cross formed of two oblique lines, intersecting like the letter X, is placed between the figures of the two factors. Thus 4×3 may be read, "4 multiplied by 3," or "4 times 3," or simply "4 into 3."

A parenthesis, ( ), consisting of two curved lines, is used to show that the numbers written within it, and connected together by the sign either plus or minus, are to be operated on alike.

Thus, $4 \times (5 + 3)$ indicates that both 5 and 3 are to be multiplied 4 times and the products added, or which gives the same result, the sum of 5 and 3 is to be multiplied by 4.

But the expression, $4 \times 5 + 3$, indicates that 5 only is to be multiplied 4 times, and 3 is to be added to the product.

So $4 \times (5 - 3)$ indicates that the difference 5 and 3 is to be multiplied 4 times, and $4 \times 5 - 3$ indicates that 3 is to be subtracted from the product 4×5.

In studying the process of multiplication it is better to consider first the case in which the multiplier is less than 10; that is, is expressed by a single figure.

Suppose it is required to multiply 243, 6 times. For convenience write the multiplier underneath the figures of the multiplicand, and draw a line under both, as follows:

$$\begin{array}{r} 243 \\ 6 \\ \hline \end{array}$$

It follows from the second principle (page 46) that we may multiply the separate parts of the multiplicand and add the partial products.

For convenience we may multiply first units, then tens, then hundreds.

That is, 6 times 3 units, 6 times 4 tens, and 6 times 2 hundreds.

Writing the figures of these partial products under each other and adding, the result is as follows:

	243
	6
Product of units....................	18
Product of tens.....................	240
Product of hundreds................	1200
Entire product.................	1458

With practice it will be found quite easy to add the partial products without writing them down. Thus,

$$\begin{array}{r} 243 \\ 6 \\ \hline 1458 \end{array}$$

and the operation may be explained by saying 6 times 3 are 18; write down 8 and reserve 1 (ten); 6 times 4 are 24, and 1 (reserved from the first partial product) are 25; write down 5 and reserve 2 (hundreds); 6 times 2 are 12, and 2 (reserved from preceding product) are 14, which is written down, and the whole result is 1458.

Whenever, then, the multiplier is expressed by a single figure the operation may thus be described: Write the figure of the multiplier under the unit figure of the multiplicand, and drawing a line underneath multiply the different orders of numbers, each one separately, beginning with the lowest. Write the products in succession, being careful to reserve the tens of any order to add as ones in the product of the next order.

When the multiplier is expressed by two or more figures the operation is still quite easy.

For example, let it be required to multiply 243 24 times.

In this case it is evident from the second principle that the multiplier may be separated into the two parts, 2 tens and 4 units, and multiplying each part separately the sum of the two products will express the entire product sought.

Writing, then, the figures of the multiplier under those of the multiplicand, so that figures of the same order shall fall in the same column, and using first the 4 units as multiplier, then the 2 tens, and adding, the operation will appear as follows:

	243
	24
First partial product, 243×4.........	972
Second partial product, 243×20......	4860
Entire product.................	5832

The first partial product is easily obtained, multiplying by 4 units; the second partial product is obtained by multiplying by 2 tens. The product of 3 units by 2 tens is obviously 6 tens; the product of 4 tens by 2 tens is obviously 8 hundreds, and the product of 2 tens into 2 hundreds is 4 thousands, and the second partial product is 4860.

Adding the two partial products together the entire product is found, 5832.

Again, to multiply 1728 by 648, the same method is easily applied as follows:

	1728
	648
First partial product.............	13824
Second partial product...........	69120
Third partial product............	1036800
Total.....................	1119744

It may be remarked that the ciphers at the right of the figures of the second and succeeding partial pro-

ducts, are usually omitted, because the omission does not affect the value of the result.

Hence the first figure of each partial product is usually written in the same column as the figure of the multiplier of that product.

Accordingly the operation of the last example would be expressed as follows:

$$\begin{array}{r@{}l} 1728 & \\ 648 & \\ \hline 13824 & \\ 6912 & \\ 10368 & \\ \hline 1119744 & \end{array}$$

In general, then, when the multiplier is composed of two or more orders of numbers, multiply by the number of each order separately, and add the partial products to obtain the entire product.

If it happens that one or more of the right hand figures of the multiplicand, or of the multiplier, or of both, are ciphers, it will be more convenient to write the multiplier so that the figure of lowest order not a cipher shall fall in the same column with the figure of the multiplicand which is of the lowest order and not a cipher.

For example, to multiply 2400 by 160, arrange the figures so that the figure 6 is found underneath the figure 4, as follows:

$$\begin{array}{l} 2400 \\ 160 \\ \hline 144 \\ 24 \\ \hline 384000 \end{array}$$

Then proceed, without regard to the ciphers, until the product is obtained, when the ciphers should be added to the right of the other figures.

It may happen that one or more ciphers are found between other figures of the multiplier, but this will cause scarcely any difficulty. For example, to multiply 2446 by 304, the operation is as follows:

$$\begin{array}{r} 2446 \\ 304 \\ \hline 9784 \\ 7338 \\ \hline 743584 \end{array}$$

It is only necessary to take care to write each partial product in its proper place; that is, place the first figure of each partial product under the figure of the partial multiplier, and the other figures in corresponding order.

It obviously follows from the foregoing that if the multiplier be 10, 100, or 1000, or any number expressed by the figure 1 with ciphers annexed, the product will be expressed by the figures of the multiplicand with the same number of ciphers annexed.

For example, if it be required to multiply 624 by 10 we may at once write the product 6240. Or to multiply 848 by 1000 we may similarly write 848000.

To verify any process of multiplication, it is usual to interchange multiplier with multiplicand.

If it happen that the two factors are equal, then the multiplier may be separated into two parts, and multiplying by each part separately, add the two partial

products together. If all the operations are correctly performed, this sum will be equal to the product first obtained.

Let it be required to multiply 982 by 982. First, by the usual operation, we have—

```
   982
   982
  ----
  1964
 7856
8838
------
964324
```

Now, to verify this result, take any number smaller than 982, as 555, and subtract from 982 to find the remaining part—

```
982
555
---
427
```

Then multiplying by each of the parts—

```
   982          982
   555          427
  ----         ----
  4910         6874
 4910         1964
4910         3928
------       ------
545010       419314
             545010
             ------
             964324
```

and adding the two results, the sum is found to be the same as before.

If 1 be multiplied any number of times, and this

product be multiplied the same number of times, and so on successively, any one of these products is called a *power* of the number used as a factor or multiplier. The power is named *first*, *second*, or *third*, and so on, according to the number of times the multiplier is used. Thus,

$1 \times 3 = 3$, the first power of 3.
$1 \times 3 \times 3 = 9$, the second power of 3.
$1 \times 3 \times 3 \times 3 = 27$, the third power of 3.
$1 \times 4 \times 4 = 16$, the second power of 4.

The second power is often called the square, and the third power of a number is called the cube. Thus 27 is the cube of 3, and 16 is the square of 4.

To indicate the power of a number, the number denoting the order or degree of the power is written above and to the right of the figures of the given number.

Thus 3^2 indicates the square of 3, and 4^3 indicates the cube of 4.

The number which indicates a power is called an exponent. The figures of exponents are usually made smaller than those above which they are placed.

EXERCISES.

(1). Multiply	328 by 2.....	*Ans.*	656.
(2). Multiply	847 by 3.....	"	2541.
(3). Multiply	20508 by 5.....	"	102540.
(4). Multiply	3605023 by 6.....	"	21630138.
(5). Multiply	9097030 by 9.....	"	81873270.
(6). Multiply	725 by 300...	"	217500.
(7). Multiply	35012 by 2000..	"	70024000.

(8). Multiply 489000 by 360...*Ans.* ....
(9). Multiply 887766 by 1000.. " ....
(10). Multiply 7198 by 216... " ....
(11). Multiply 7575 by 7575.. " ....
(12). Multiply 871836 by 418... " ....
(13). Multiply 671834 by 871... " ....
(14). Multiply 871387 by 834... " ....
(15). Multiply 91864 by 913... " ....
(16). Multiply 18374 by 944... " ....
(17). Find the value of
$18 \times (24+36-10+40)$. *Ans.* 1620.
(18). Find the value of
$36 \times (24+36-10)+40$. *Ans.* 940.
(19). Find the value of
$2 \times 3 \times 4 \times (12-15+30-4+20)$. *Ans.* ....
(20). Find the value of
$(5 \times 6+20) \times (48+12 \times 5)$. *Ans.* ....

REVIEW VI.

a. **Multiplication is a process** of counting a number of things a number of times together.

b. **The multiplicand** is the number of things to be counted.

c. **The multiplier** is the number denoting how many times the multiplicand is to be counted.

d. **The product** is the result obtained by multiplying.

e. **The multiplier may be** one, or more than one, or less than one. According as the multiplier is more or less than one, **the product is more or less** than the multiplicand.

f. To "**multiply by**" a number, means to multiply the multiplicand that number of times.

g. The multiplicand and multiplier are often called **the factors** of the product, and in general a factor of a number is any one of the integral numbers which, multiplied together, will produce that number.

h. **The product of the sum of any two numbers** by any multiplier, is equal to the sum of the products of the separate numbers multiplied by the same multiplier.

i. **The product of any number by the sum of two numbers,** is equal to the sum of the products of the same multiplicand multiplied by the same two numbers separately.

j. **The product of the difference** of two numbers multiplied by any number, is equal to the difference of the products of the same numbers multiplied by the same multiplier.

k. The product of any number **into the difference of two numbers,** is equal to the difference of the products of the same multiplicand multiplied into the numbers separately.

l. **The product of any two numbers is the same,** whichever is taken as the multiplicand, the other being taken as the multiplier.

m. **To multiply by the product of any two or more factors,** gives the same result as to multiply successively by the separate factors.

n. **The sign of multiplication** is a cross formed by two oblique lines, intersecting in the form of the letter X.

o. **A parenthesis is used** to show that the numbers whose symbols are contained within it, and connected by the sign + or –, are to be operated on alike.

p. **When the multiplier is less than ten,** the multiplication is performed by multiplying the numbers of different orders in the multiplicand, each one separately, beginning with the lowest, and writing the products in order,

taking care to reserve tens of any one order to count as ones of the next higher, as in addition.

q. **When the multiplier is more than ten,** that is, composed of two or more orders of numbers, the multiplication is performed by multiplying by the number of each order separately, and adding these partial products to obtain the entire product.

r. **If the multiplier is expressed by the figure 1 with ciphers annexed,** as 10, 100, 1000, and so on, the product will be expressed by the figures of the multiplicand with the same number of ciphers annexed.

s. **A power of a number,** is a product obtained by multiplying **one** by that number, any number of times successively, the degree of the power being named first, second, third, or fourth, and so on, according as the number is used as a factor one, two, three, or four, or more times.

t. **The second power** is also called the square, and the third power is called the cube.

u. **The exponent of a power,** is the number

indicating the order or degree of the power, written above and to the right of the figure (or figures) of the number whose power is denoted, and written in smaller figures.

v. **Multiplication may be verified** by interchanging multiplicand for multiplier. It may also be verified by separating the multiplier into two parts, multiplying by each part separately, then adding these two partial products, whose sum should equal the original product.

CHAPTER VII.

Division, or Finding a Factor.

It is often required to find how many times one number must be multiplied to produce another. This process is called division. For example, 24 is how many times 6? This is easily answered, for it is remembered that 2 times 6 are 12, 3 times 6 are 18, 4 times 6 are 24, and 4 is the answer sought.

The question might have been in the form, What number must be multiplied 6 times, to produce 24? In this case, the answer would still be the same, and in either case we should have a product and one factor given, to find the other factor.

The given number which equals the product of another given number multiplied into a number sought, is called the *dividend*. The given number whose product, when multiplied into a number sought, must equal the dividend, is called the *divisor*. The number sought which multiplied into the divisor must produce the dividend, is called the *quotient*. Thus, in the example last given, 24 is the dividend, 6 is the divisor, and 4, the number sought, is the quotient.

According to the usage of language, the dividend is said to be *divided by* the divisor, or the divisor is said to be *divided into* the dividend, in order to find the quotient.

To indicate that one number is to be divided by another, a sign is formed by placing a short horizontal line between two dots, one vertically over the other, thus, ÷. The figures of the dividend precede, those of the divisor follow it. Division may also be indicated by placing the figures of the dividend above, those of the divisor below a horizontal line. Thus, $24 \div 6 = 4$, or $\frac{24}{6} = 4$, is read 24 divided by 6 equals 4.

So long as the dividend and divisor are both small numbers, it is easy to find the quotient at once by a few trials, as in the example already given, but in the case of larger numbers, it becomes necessary to follow a systematic method, which will now be explained.

According to the definition given, the product of the quotient, multiplied into the divisor, equals the dividend. Hence it follows, if the quotient be separated into any number of parts, and each be multiplied by the divisor, the sum of the products would equal the dividend.

Hence, if one of these partial products be subtracted from the dividend, the remainder will equal the sum of the remaining partial products.

If from this remainder another partial product be subtracted, and the process be continued until all the partial products are successively subtracted, there will then be no remainder.

For illustration, let it be required to divide 2616 by 6. For this purpose, write the dividend and divisor in any convenient manner. Usually, however, the figures of the divisor are written at the left of those of

the dividend, a short curve placed between, to separate them.

Another curve at the right indicates a place for the figures of the quotient. Thus:

$$6)2616($$

Next let us find the highest order of number that may be contained in the quotient. The dividend contains thousands, but the quotient must be less than 1000, because this number, multiplied by the divisor 6, gives a product 6000, which is more than the dividend. If now we try the order of hundreds, and multiply 100 by 6, the product is 600, which is less than the dividend, and so the quotient is more than 100. If we try in succession, 200, 300, 400, and 500, we find the quotient must be more than 400 and less than 500.

Let us consider then 400 as a partial quotient, and subtract the partial product (or 6×400), from the dividend, and we have the first remainder, 216. Thus:

$$\begin{array}{rcl} & 6)2616(400+30+6 & \\ 6 \times 400 = & \underline{2400} & = \text{First partial product.} \\ \text{First remainder} = & 216 & \\ 6 \times 30 = & \underline{180} & = \text{Second partial product.} \\ \text{Second remainder} = & 36 & \\ 6 \times 6 = & \underline{36} & = \text{Third partial product.} \\ \text{Third remainder} = & 0 & \end{array}$$

But this remainder, 216, must equal the product of the divisor 6 into the remaining part of the quotient, and we may treat 216 as a new or partial dividend.

Proceeding as before, we find the quotient of this second dividend must be more than 30 and less than 40.

Take 30 as a partial quotient, multiply it by the divisor 6, and subtracting the partial product 180 from the partial dividend, we have the second remainder 36. Treating this as a third partial dividend, the third partial quotient 6 is easily obtained, and subtracting the product of this quotient multiplied into the divisor, 36, the third remainder is zero, or there is no remainder, and the sum of the partial quotients is the entire quotient sought.

For it is obvious that we have the true quotient if, being multiplied by the divisor, it produces the dividend. But the sum of the products of the divisor multiplied into the partial quotients, is equal to the dividend, because these products have been subtracted successively, and nothing remained. Hence the product of the divisor multiplied into the sum of the partial quotients—that is, into the entire quotient—must also equal the dividend.

The method itself may be described as follows:

Write the figures of the dividend and divisor in convenient places. Find by trial the highest number of the highest order that may be contained in the quotient. Consider the number thus found a part of the quotient (or a partial quotient), multiply it into the divisor, and subtract the product from the dividend. Use the remainder thus obtained as a new dividend, and proceed as with the first or original dividend; that is, find the highest number of the next lower order that may be contained in the second partial quotient.

e. **The multiplier may be** one, or more than one, or less than one. According as the multiplier is more or less than one, **the product is more or less** than the multiplicand.

f. To "**multiply by**" a number, means to multiply the multiplicand that number of times.

g. The multiplicand and multiplier are often called **the factors** of the product, and in general a factor of a number is any one of the integral numbers which, multiplied together, will produce that number.

h. **The product of the sum of any two numbers** by any multiplier, is equal to the sum of the products of the separate numbers multiplied by the same multiplier.

i. **The product of any number by the sum of two numbers,** is equal to the sum of the products of the same multiplicand multiplied by the same two numbers separately.

j. **The product of the difference** of two numbers multiplied by any number, is equal to the difference of the products of the same numbers multiplied by the same multiplier.

k. The product of any number **into the difference of two numbers,** is equal to the difference of the products of the same multiplicand multiplied into the numbers separately.

l. **The product of any two numbers is the same,** whichever is taken as the multiplicand, the other being taken as the multiplier.

m. **To multiply by the product of any two or more factors,** gives the same result as to multiply successively by the separate factors.

n. **The sign of multiplication** is a cross formed by two oblique lines, intersecting in the form of the letter X.

o. **A parenthesis is used** to show that the numbers whose symbols are contained within it, and connected by the sign + or –, are to be operated on alike.

p. **When the multiplier is less than ten,** the multiplication is performed by multiplying the numbers of different orders in the multiplicand, each one separately, beginning with the lowest, and writing the products in order,

the multiplications and subtractions being carried on mentally, the method is called ***short division***.

When the processes of multiplication and division, which form a part of the operation of division, are expressed, the method is called ***long division.***

It may happen that there is a remainder after subtracting the final product from the last partial dividend.

For example, dividing 4 into 275, we have the quotient 68, and remainder 3, as follows:

$$4\underline{)275}$$
$$68 \text{ —, } 3 \text{ remainder.}$$

Now 275 is the sum of 272 and 3, and the quotient of 275, divided by 4, must equal the sum of the quotients of 272 and 3 each divided by 4. But the quotient of 272, divided by 4, is 68, and it only remains to express the quotient of 3, divided by 4. This is indicated by writing the dividend 3 above and the divisor 4 below a horizontal line, thus, $\frac{3}{4}$; and the entire quotient of 275, divided by 4, is therefore written $68\frac{3}{4}$, and read *sixty-eight and three-fourths*, or *sixty-eight and three divided by four*.

It will be observed that this quotient of 3 divided by 4 is expressed as a fractional number, and it will be found that *any fractional number may be regarded as the quotient of the numerator divided by the denominator*.

Division is said to be *exact* when the quotient is an exact whole number.

Before leaving this part of the subject it will be

well to notice some examples of division where the divisor is a large number.

The method is the same as first explained, yet the learner may find difficulties for which a few suggestions will be useful.

Let it be required, then, to divide 4468216 by 583. The operation, as written, will appear as follows:

$$\begin{array}{rl}
 & 583)4468216(7664\tfrac{104}{583} \\
583\times 7 = & \underline{4081} \\
 & 3872 \\
583\times 6 = & \underline{3498} \\
 & 3741 \\
583\times 6 = & \underline{3498} \\
 & 2436 \\
583\times 4 = & \underline{2332} \\
 & 104 = \text{remainder.}
\end{array}$$

It is easily perceived that the highest order of number in the quotient must be that of *thousands*. By successive trials the first partial quotient is found to be 7 (thousands), and multiplying into the divisor, and subtracting, the remainder 387 (thousands) is found, and uniting 2 hundreds with it, the second partial dividend is 3872 (hundreds). In a similar way the second partial quotient 6 (hundreds) is found, and so on.

But it is probable the learner would need to make a number of trials before obtaining the first partial quotient 7 (thousands), or the second partial quotient 6 (hundreds), and so on.

If, however, 6 (hundred) be used as a trial divisor (because 583 is nearer 6 hundred than 5 hundred), and 44, the number expressed by the first two figures of the first partial dividend, be taken as trial dividend, the first quotient, figure 7, is easily obtained; for 6 divided into 44 gives the quotient 7, with a remainder. In a similar way, taking 38, the number expressed by the first two figures of the second partial dividend, as a trial dividend, and using the same trial divisor, the second quotient figure 6 is easily obtained.

Again, taking the number expressed by the first two figures of the third partial dividend, as a third trial dividend, with the same trial divisor, the third quotient figure 6 is also easily found, and in like manner the next quotient figure 4.

As another example, let it be required to divide 3898556 by 4183.

```
4183)3898556(932
     37647
     -----
      13385
      12549
      -----
        8366
        8366
        ----
```

In this case we take 4 (thousand) as the trial divisor, and for the first trial dividend use 38 (hundred thousand), and the quotient of 4 divided into 38 is easily found to be 9 (hundred).

Of the second dividend, using 13 (ten thousands) as a trial dividend, the quotient 3 (tens) is found.

Finally, using 8 (thousands) in the third dividend in a similar way, the partial quotient 2 is found.

In any case we may use as a trial divisor the number of highest order only (in the true divisor), and for a trial dividend use the number of the highest order, or the two highest orders, of the true dividend, as may be needed. It will sometimes happen that the partial quotient indicated by this method will be too large or too small, but practice will soon enable the learner to estimate each partial quotient readily.

It is important to bear in mind, in writing the figures of the quotient, that a cipher must be written when no other figure expresses the number of any order.

For illustration, let it be required to divide 76125 by 25.

```
25)76125(3045
   75
   ---
    112
    100
    ---
     125
     125
     ---
```

In this case the first partial quotient is 3 (thousands) and the first partial remainder is 1 (thousand), or taking the hundreds with it for a new partial dividend, this becomes 11 hundred. But 25 divided into 11 hundreds does not give a quotient of the order of hundreds, therefore write a cipher in the hundred's place in the figures of the quotient. By using the 2 tens of the dividend the partial dividend becomes 112 tens, and

the partial quotient, 4 tens, is easily found, and the remaining part of the process is completed without difficulty.

It was found that to multiply by 10 or by 100, or by any number expressed by the figure 1 with ciphers annexed, it was sufficient to annex the same number of ciphers to the figures of the multiplicand. It follows, therefore, that to divide by 10 or 100 (or by any power of 10), it is sufficient, to express the quotient, to omit from the figures of the dividend the same number of ciphers at the right, if they are found there. In case they are not there, it may easily be shown that the same number of figures may be omitted, and that the number expressed by them is in fact a remainder.

Thus, 270 divided by 10 gives the quotient 27, and 1500 divided by 100 gives the quotient 15. Or 15000 divided by 100 gives the quotient 150. Again, 1525 divided by 100 gives the quotient 15, with a remainder 25; or the entire quotient may be expressed as $15\frac{25}{100}$.

The method of division, we have seen, is based on the principle that the entire quotient must be equal to the sum of the partial quotients obtained by dividing the several parts of the dividend. And in general it is true that the quotient of the sum of two numbers is equal to the sum of the quotients of the respective numbers, the divisor being the same. Thus $\frac{96+72}{24}=\frac{168}{24}=7$. But $\frac{96}{24}=4$, and $\frac{72}{24}=3$, and $3+4=7$.

That is to say, the quotient of 168 (the sum of 96 and 72), divided by 24, is 7, which is the sum of 7 and

3, the quotients of the respective numbers, 96 and 72, each divided by 24.

Again, since the product of the difference of two numbers is equal to the difference of the products of the respective numbers (the multiplier being the same), it follows that the quotient of the difference of two numbers is equal to the difference of the quotients of the respective numbers—each divided by the same divisor.

Thus, for illustration, it is clear that if $7 = 4 + 3$, then $4 = 7 - 3$. It is as evident that if $\frac{168}{24} = \frac{96}{24} + \frac{72}{24}$, then $\frac{96}{24} = \frac{168}{24} - \frac{72}{24}$.

But 96 is the difference between 168 and 72, and the quotient of this difference is equal to the difference of the quotients, as before stated. The same reasoning would evidently apply in any similar case.

Hence it follows if each of two numbers is exactly divisible by any divisor, their difference is exactly divisible by the divisor.

Further, it was found that to multiply by the product of two or more factors, amounts to the same as to multiply successively by the same factors. Hence, to divide by the product of two or more divisors, amounts to the same as to divide successively by the divisors.

For instance, if $24 \times 2 \times 3 = 24 \times 6 = 144$, then it obviously follows that $144 \div 6 = (144 \div 2) \div 3$; that is, to divide 144 by 6 (which is the product of 2×3) gives the same result as to divide 144 by 2, and that quotient again by 3.

Hence, any equal factors of dividend and divisor

may be omitted without affecting the value of the quotient. The omitting of equal factors from dividend and divisor is called *cancelling.*

The principle of cancelling is frequently useful in abbreviating the operations of multiplication and division, especially where the dividend and divisor are expressed as indicated products of several factors.

Suppose we have the expression $\frac{12\times 8\times 7}{4\times 6}$, where the dividend is written above the line and the divisor below.

If the factors of the dividend are multiplied together the result is 672, and the factors of the divisor multiplied together give 24. Performing the division the quotient 28 is found.

Again writing the expression $\frac{12\times 8\times 7}{4\times 6}$, divide both dividend and divisor by the factors 4 and 6 and the result may be indicated as follows:

$$\frac{\cancel{12}^{2}\times \cancel{8}^{2}\times 7}{\cancel{4}\times \cancel{6}^{1}}$$

That is, dividing by 4, we may divide any one of the factors of the dividend, say 8, and write the quotient figure 2 above and near the figure 8.

Dividing into the factor 4 of the divisor, the quotient 1 is not usually expressed, but understood. Again dividing 6 into the factor 12 of the dividend, and into the factor 6 of the divisor, the results are indicated in a similar manner.

The factors of the dividend remaining are now

$2 \times 2 \times 7 = 28$, and the divisor becomes 1; that is $\frac{28}{1} = 28$, and the quotient 28 is the same as first obtained. With this explanation, the following example will be easily understood:

$$\frac{\cancel{8}^2 \times \cancel{12}^2 \times \cancel{15}^5 \times \cancel{21}^3}{\cancel{8} \times \cancel{6} \times \cancel{4} \times \cancel{7}} = \frac{2 \times 2 \times 5 \times 3}{1} = 60.$$

Or this,

$$\frac{\cancel{9} \times \cancel{12}^3 \times 27}{\cancel{4} \times \cancel{36}^4} = \frac{3 \times 27}{4} = \frac{81}{4} = 20\tfrac{1}{4}.$$

In these examples much of the labor of multiplying and dividing has been avoided by means of cancelling.

To verify any process of division, multiply the divisor and quotient together, and the product should equal the dividend.

EXERCISES.

(1).	Divide	656 by 2......	*Ans.*	328.
(2).	Divide	2541 by 3......	"	847.
(3).	Divide	102540 by 5......	"	20508.
(4).	Divide	21630138 by 6......	"	3605023.
(5).	Divide	81873235 by 9......	"	$9097030\frac{5}{9}$.
(6).	Divide	217500 by 300....	"	725.
(7).	Divide	70024000 by 2000...	"	35012.
(8).	Divide	1203033 by 3679...	"	327.
(9).	Divide	49561766 by 5137...	"	9648.
(10).	Divide	2150596762 by 125....	"	$17204774\frac{12}{125}$.
(11).	Divide	78674 by 200....	"	$393\frac{74}{200}$.

(12). Divide 71900715708 by 57149.

Ans. $1258127\frac{15785}{57149}$.

(13). Find the value of $(123 + 47 - 30 + 84) \div 6$.

(14). Find the value of $(480-243+123) \div 24$. *Ans.* 15.

(15). Divide 1728 by 3×8. *Ans.* 72.

(16). Find the value of $\frac{8 \times 21 \times 30 \times 16 \times 3}{4 \times 9 \times 8 \times 3}$. *Ans.* 40.

(17). Divide $16 \times 5 \times 14 \times 20 \times 32 \times 30 \times 50$ by $40 \times 24 \times 50 \times 20 \times 7 \times 10$. *Ans.* 16.

(18). Divide $213 \times 84 \times 190 \times 264$ by $30 \times 56 \times 36$. *Ans.* ..

(19). Find the value of $\frac{28 \times 60 \times 56 \times 64}{49 \times 32 \times 40 \times 8}$. *Ans.* ..

(20). Find the quotient of $\frac{108 \times 256}{45}$. *Ans.* ..

(21). Find the quotient of $\frac{859}{24}$.

REVIEW VII.

a. **Division** is the process of finding how many times one number must be multiplied to produce another.

b. **The dividend** is that given number which equals the product of another given number multiplied into a number sought.

c. **The divisor** is that given number, the product of which multiplied into a number sought, equals the dividend.

d. **The quotient** is the number sought which, multiplied into the divisor, produces the dividend.

e. **The dividend** is said to be divided by the divisor, or the divisor is said to be divided into the dividend.

f. **The symbol of division** is formed by placing a short horizontal line between two dots, one vertically over the other, as follows: ÷. The figures of the dividend are placed before, those of the divisor after it.

Division may also be indicated by placing the figures of the dividend above, those of the divisor below, a horizontal line.

g. **The dividend may be separated into several parts or partial dividends,** each of which may be divided by the given divisor, and the sum of the partial quotients thus obtained will equal the entire quotient sought. Because the product of the divisor into the entire quotient must equal the sum of the products of the divisor into each of the partial quotients.

h. **The method of division,** based on the

principle just stated, is as follows: Having written the figures of the dividend and divisor in convenient places, find by trial the highest number of the highest order that may be contained in the quotient. Consider this number a part of the quotient, multiply it into the divisor, and subtract the product from the dividend. Use the remainder thus obtained as a new partial dividend, and proceeding as before find the highest number of the next lowest order that may be contained in this partial quotient. So continue until there is no remainder, or until the remainder is less than the divisor. In case of such a remainder, express the partial quotient by writing the figures of the remainder above, and those of the divisor below, a horizontal line. The sum of these partial quotients is the entire quotient sought.

A division is said to be exact when the quotient is an exact whole number.

i. **When the divisor is a small number,** say less than 10, the necessary multiplications and subtractions may be carried on mentally, writing down at once the figures of the quo-

tient. In this case the process is called **short division.**

When the processes of multiplication and subtraction belonging to the operations of division are expressed, the method is then called **long division.**

j. **In case the divisor is some power of** 10, that is, expressed by the figure 1 with ciphers annexed, the quotient may be expressed by writing the figures of the dividend, omitting as many of the right-hand figures as the ciphers in the figures in the divisor, the number expressed by the figures omitted being considered a remainder.

k. **The quotient of the sum of two numbers** is equal to the sum of the quotients of the respective numbers, the divisor being the same.

l. **The quotient of the difference of two numbers,** is equal to the difference of the quotients of the respective numbers, the divisor remaining the same.

Hence if each of two numbers be exactly divisible by any divisor, their difference is exactly divisible by the same divisor.

m. **To divide by the product of two or more divisors** gives the same result as to divide first by one divisor and then divide the resulting quotient by the other divisor.

Hence, any equal factors of dividend and divisor may be omitted without affecting the value of the quotient. The omitting of equal factors from dividend and divisor is called **cancelling.**

n. **To verify any process of division,** multiply the divisor and quotient together, and their product should equal the dividend.

CHAPTER VIII.

Of Multiples, Common Multiples, Common Factors, and Greatest Common Factors.

Any fact which may be affirmed of a number, is a property of that number.

If it happen that some property of frequent use belongs to a large series of numbers, a name is often given to distinguish such a class.

Thus a number exactly divisible by the factor 2 is called *even*, and any number not exactly divisible by 2 is called an odd number.

Any number is exactly divisible by itself and by one; but a number which is exactly divisible by no other integral factor than itself and one is called a prime number. A number exactly divisible by some integral factor other than itself and one is called a composite number, in other words, any number not prime.

It is easily found that 4, 6, 8, 25, and so on, are composite numbers, and that 1, 2, 3, 5, 13, 29, and so on, are prime numbers.

A multiple of a number is a product obtained by multiplying that number by any integral factor.

Thus, 8, 12, 16, and 20, are multiples of 4; and 9, 18, and 27, are multiples of 3.

A *common multiple* of two or more numbers is at the same time a multiple of each of them. Thus 12

is a common multiple of 4 and 6, and 24 is a common multiple of 3, 4, 6, and 12.

The least common multiple of two or more numbers is the least of all the multiples common to those numbers. Thus, 24 is a common multiple of 4 and 6; so is 36, but it is easily shown that 12 is the least common multiple of the numbers 4 and 6.

Since a multiple of a number is some product of that number, it must be a multiple of each prime factor of that number, in case it is composite, and hence any common multiple of two or more numbers must be a multiple of each and every prime factor to be found in those numbers.

As some of the numbers may contain the same factor several times, it follows that the common multiple, and of course the least common multiple, must contain each prime factor as many times as it is found in any one of the given numbers.

If then a product be formed by taking, as factors, each prime factor to be found in the given numbers, using each factor only as many times as the greatest number in which it is found in any of the numbers, then such product must be a common multiple, and the least common multiple of those numbers.

Suppose it were required to find the least common multiple of 8 and 10. The factors of 8 are $2 \times 2 \times 2$, and the factors of 10 are 2×5. Now whatever multiple of 8 is considered, it must contain the factor 2 taken three times. Whatever multiple of 10 is considered must contain the factor 5 once, and the factor 2 once. But the multiple of 8 must contain the factor 2 more than once.

It is then easy to see that the product formed by the factors $5 \times 2 \times 2 \times 2$, must be a multiple of 10, and must be a multiple of 8, since it contains the factors of each, and it is also obvious that it is the least common multiple of the two numbers.

In general, then, to find the least common multiple of several numbers, it is necessary to find the prime factors of each of the several numbers, and form a product by using each of the prime factors as many (and only as many) times as the greatest number of times it is found in any of the given numbers.

Thus, to find the least common multiple of 15, 20, and 25.

The prime factors may be indicated as follows:

$$\begin{aligned} 15 &= 5 \times 3 \\ 20 &= 5 \times 4 \\ 25 &= 5 \times 5 \end{aligned}$$

and it appears from inspection, that 3 is a factor only once, 4 only once, and 5 appears twice as a factor in 25.

Form the product, $3 \times 4 \times 5 \times 5 = 300$, and 300 is the least common multiple of the numbers 15, 20, and 25.

It may be asked how the prime factors of large numbers are to be found, and to this question it may be answered, that in any case, these may be found by dividing by the prime numbers, one after another, beginning with the lowest in order, as, 2, 3, 5, etc., until some factor is found, if such there be. For example, to find the factors of 231.

First, it is obvious that 2 is not a factor. If then 3 be tried, it gives the quotient 77. If 77 be exam-

ined for prime factors, it is easy to see that 3 is not a factor, neither is 5, but 7 is at once recognized, and 11, because $7 \times 11 = 77$.

The factors of 231 are, then, 3, 7, and 11.

Again, let it be required to find the prime factors of 2431. Trying the divisors 3, 5, and 7, one after another, no integral quotient is obtained, but dividing by the next higher prime number, that is 11, the quotient is 221.

$$11\overline{)2431}$$
$$221$$

Then 11 is a factor, and in seeking another factor, no number smaller than 11 need be tried, because we have already found that such numbers are not factors in the original number, 2431. But we can only tell by trial whether 11 is not again a factor in 221. By trial, however, it is found not to be one.

Dividing by 13, the quotient is exactly 17, which is at once recognized as a prime number. The factors of 2431 are thus found to be $11 \times 13 \times 17$.

Common Factors.

We have learned that a factor of a number is one of two or more numbers that multiplied together will produce that number.

A common factor of two or more numbers is at the same time a factor of each. Thus 4 is a common factor of 8, 12, and 20, because it is a factor of each.

The greatest common factor is obviously the greatest factor common to several numbers. Thus 3 is a

common factor of 36 and 48, so is 6, but 12 is found to be the greatest common factor of those two numbers.

If it is required to find the greatest common factor of two or more numbers, it is obvious that were the numbers separated into their prime factors, a simple inspection would enable one to select all the common factors. It is also evident that the product of all the common factors will be the greatest common factor, since it contains all the common factors.

Suppose it were required to find the greatest common factor of 30, 42, and 60.

The prime factors of each may be indicated as follows:

$$30 = 2 \times 3 \times 5$$
$$42 = 2 \times 3 \times 7$$
$$60 = 2 \times 2 \times 3 \times 5$$

It is seen at once that the factors 2 and 3, are common to the three given numbers, and that no other factor is.

Each of the given numbers must be divisible by the product 2×3, that is by six. No number greater than 6 can divide each one exactly, because there would then be some other common factor. Hence, 6 is the greatest common factor.

In any case, then, when numbers are separated into their prime factors, the greatest common factor may be found by multiplying together the factors that are common.

In the case of large numbers it is not always convenient to find the prime factors, and for such cases another method of finding the greatest factor has been devised.

This method is based upon the principle that any common factor of the dividend and divisor is a factor of the remainder, if there be any when the division is performed.

When a remainder is left, in case of division, it is the difference between the dividend and the product of the divisor multiplied into the integral part of the quotient. But any such product of the divisor will be a multiple of the divisor.

It follows, then, that the remainder in this case is the difference between the dividend and some multiple of the divisor.

It follows, then (from VII., *l*), that this remainder is divisible by the greatest common factor sought.

We are now ready to understand the other method of finding the greatest common factor of any two numbers.

Having divided the larger number by the smaller, there will be a remainder, unless the smaller number is itself the greatest common factor. It is then sufficient to find the greatest common factor of this remainder and the divisor. To do this we may use the divisor as a new dividend, and the remainder as a divisor. In case there is a second remainder we may again repeat the preceding operation, and continue until there is an exact division with no remainder. It is evident that the last divisor will be the greatest common factor of itself and the preceding divisor (or the last dividend). It must, therefore, be the greatest common factor of the preceding dividend and divisor, and finally of the first dividend and divisor.

If it happens that the greatest common factor is 1,

the numbers are said to be prime to each other. Thus 4 and 9 are prime to each other.

Suppose it were required to find the greatest common factor of 264 and 768.

According to the foregoing method the operation will be as follows:

```
264)768(2
    528
    ---
    240)264(1
        240
        ---
         24)240(10
            240
            ---
```

It thus appears that 24 is the greatest common factor sought.

Again, to find the greatest common factor of 706 and 3505. The operation is as follows:

```
706)3505(4
    2824
    ----
     681)706(1
         681
         ---
          25)681(27
             50
             ---
             181
             175
             ---
               6)25(4
                 24
                 --
                  1)6(6
                    6
                    -
```

In this case 1 is the greatest common factor, and the numbers are therefore prime to each other.

If it is required to find the greatest common factor of three or more numbers, first find it for two, then for that factor and the third number, and so on.

For example, to find the greatest common factor of 442, 612, and 697.

First find the greatest common factor of 442, and 612, as follows:

```
442)612(1
    442
    ---
    170)442(2
        340
        ---
        102)170(1
            102
            ---
             68)102(1
                 68
                 --
                 34)68(2
                    68
                    --
                     0
```

This, it appears, is 34, and we now find the greatest common factor of 34 and 697 as follows:

```
34)697(20
   68
   --
    17)34(2
       34
       --
        0
```

And 17 is the greatest common factor of the three numbers, 442, 612, and 697.

EXERCISES.

(1). Find the prime factors of 285. *Ans.* $3 \times 5 \times 19$.

(2). Find the prime factors of 608.
Ans. $2 \times 2 \times 2 \times 2 \times 2 \times 19$.

(3). Find the prime factors of 728. *Ans.*

(4). Find the prime factors of 2808. *Ans.*

(5). Find the prime factors of 719. *Ans.*

REMARK.—If there are any integral factors of 719, one at least, must be less than 27, because $27 \times 27 = 729$.

(6). Find the least common multiple of 25, 35, and 45.
Ans. 1575.

(7). Find the least common multiple of 64, 84, and 132.
Ans. 14784.

(8). Find the least common multiple of 16, 28, and 84.
Ans. 336.

(9). Find the least common multiple of 25, 31, and 50.
Ans. 1550.

(10). Find the least common multiple of 12, 15, 20, and 25. *Ans.* 300.

(11). Find the greatest common factor of 336 and 812.
Ans. 28.

(12). Find the greatest common factor of 4082 and 8476.
Ans. 26.

(13). Find the greatest common factor of 84, 156, and 276. *Ans.* 12.

(14). Find the greatest common factor of 141, 799, and 940. *Ans.* 47.

(15). Find the greatest common factor of 4096 and 6528. *Ans.*

REVIEW VIII.

a. **A composite number** is exactly divisible by some integral number other than itself and one, or a composite number is the product of two or more integral factors, each greater than one.

b. **A prime number** is not exactly divisible by any integral number besides itself and one.

c. **A multiple of a number** is a product of that number multiplied by any integral multiplier.

d. **A common multiple of two or more numbers** is at the same time a multiple of each of them, and the **least common multiple** is the least of such common multiples.

e. **To find the least common multiple of two or more numbers** first find the prime factors of each number, then form a product by using each prime factor as many (and only as many) times as the greatest number of times it is found in any of the given numbers.

f. **To find the prime factors,** when these are not obvious from inspection, divide by the

prime numbers, one after another, until an exact divisor is found, or until it appears the number itself is prime.

g. **A common factor of two or more numbers** is a factor of each of the numbers, and the greatest common factor is the greatest factor common to several numbers.

h. **The greatest common factor** is the product of all the prime factors common to several numbers.

i. One method of finding the greatest common factor of several numbers consists in finding the prime factors of each number and then forming the product of all that are common.

j. Another method of finding the greatest common factor of several numbers depends upon the principle that any common factor (and, therefore, the greatest common factor) of a dividend and divisor is a factor of the remainder, if there be any, when the division is performed.

k. **To find the greatest common factor** divide the larger number by the smaller. (If

there be no remainder the smaller number is itself the greatest common factor.) If there be a remainder, use it as a new divisor, and use the first divisor as a new dividend, and so continue until an exact divisor is found. This will be the greatest common factor sought.

l. If the greatest common factor is one, the numbers are prime to each other.

m. When three or more numbers are given, the greatest common factor of any two may be found, and then the greatest factor common to this factor and a third of the given numbers, and so on.

CHAPTER IX.

Fractional Numbers.

It has already been stated:

1st. A fractional number denotes some portion of a thing.

2d. A fractional number is expressed as some number of equal parts into which a thing is supposed to be separated.

3d. A fractional number is named by uniting the name of one of the equal parts to that of the number of the equal parts considered.

The number of equal parts into which a thing is supposed to be separated is called the *denominator*, and the number of the equal parts used is called the *numerator*.

Thus three-fourths is a fractional number which denotes that a thing is separated into four equal parts (each one called a fourth), and that three of these equal parts are used. Four is the denominator and three is the numerator.

It has also been stated that a fractional number is expressed in the Arabic notation by writing the figures of the numerator above and the figures of the denominator below a line usually horizontal, but which may be oblique. Thus three-fourths is expressed in figures

$\frac{3}{4}$ or $^3/_4$, and seven-fifths, $\frac{7}{5}$ or $^7/_5$, each expression being read the same as the words for which they stand.

The numerator and denominator are called the *terms* of a fractional number.

The written (or printed) expression of a fractional number, according to the Arabic notation, is called a *fraction.* Very often the name fraction is given to the fractional number itself; but to avoid confusion, and to keep clearly before the mind the distinction between a fractional number and the figures which express it, the term fraction will be used in this treatise only for *expressions* which represent fractional numbers or indicate divisions.

Thus the expression $^3/_4$ is called a fraction, and it represents the fractional number three-fourths.

The expression $\frac{4\frac{1}{2}}{6\frac{2}{3}}$ is also called a fraction, and it represents an indicated division. It is read "$4\frac{1}{2}$ divided by $6\frac{2}{3}$." How this division may be performed will be seen further on.

A fraction of which either term is a mixed number, is called a *complex fraction.*

The terms of a fractional number are also considered the terms of the fraction which represent it.

As a single whole dollar is equal in value to four quarters of a dollar, so in the same sense the value of the number four-fourths is equal to one, or in Arabic notation, $\frac{4}{4} = 1$. In a similar way $\frac{5}{5} = 1$, or, in fact, any fractional number whose numerator and denominator are equal, is equal to 1, because anything is equal to the sum of all its parts.

In like manner, if the numerator is less than the denominator, the fractional number is less than one, because a part is less than the whole.

If the numerator is greater than the denominator, the value of the number is more than one, because there are more parts indicated than is required to make a single whole thing.

A fraction or fractional number, whose value is less than 1, is called proper.

A fraction or fractional number, whose value equals or exceeds 1, is called improper.

Thus $\frac{4}{4}$, $\frac{5}{3}$, $\frac{11}{7}$, $\frac{12}{6}$ are improper fractions, and $\frac{3}{4}$, $\frac{3}{5}$, $\frac{7}{11}$, $\frac{6}{12}$ are proper fractions.

A number expressed in two parts, one integral the other fractional, is called a *mixed number*.

Thus $4\frac{1}{2}$ (read, "four and a half") and $5\frac{2}{3}$ (read, "five and two-thirds") are mixed numbers.

It has been observed that a fraction may be understood to express either a fractional number or an indicated division. Thus $\frac{2}{3}$ may be read either "two-thirds" or "two divided by three." The phrase *two-thirds* indicates that a thing is separated into three equal parts, and that two of these parts are taken, while the phrase "two divided by three" indicates two things are divided by three. It is obvious that the value of the result is the same in either case, but the operations are not identical

REVIEW IX.

a. **A fractional number** is one which denotes a portion of a thing.

b. **A fractional number** is expressed as some number of the equal parts of a thing.

c. **The number of the equal parts** into which a thing is supposed to be separated is called the denominator of a fractional number.

d. **The number of the equal parts considered** is called the numerator of a fractional number.

e. **A fractional number** is expressed in the Arabic notation by writing the figures of the numerator above, and those of the denominator below, a line either horizontal or oblique.

f. **The written expression** which represents a fractional number is called a **fraction.**

A fraction is also used to indicate an unperformed division.

g. **A proper fraction** is one whose numerator is less than the denominator.

h. **An improper fraction** is one whose numerator equals or exceeds the denominator.

i. **The numerical value** of a proper fraction is less than 1, that of an improper fraction is 1, or more than 1.

j. **A mixed number** is expressed in two parts, one in the integral form, the other in the fractional form.

CHAPTER X.

Reduction of Fractions, or Changing the Form without Changing the Value.

All the operations that have been applied to integral numbers—that is, addition, subtraction, multiplication, and division—may be applied to fractional numbers, and when so applied are really the same in their nature, but the processes are not as simple, and an explanation of these will be needed by the learner.

First, however, it may be remarked that some change in the form of a fraction is often necessary to prepare it for a required operation. Any change in the form of a fraction that does not affect its value is called a *reduction of the fraction.*

First Reduction.

The most usual change in the form of a fraction consists in multiplying or dividing the numerator and denominator by the same factor.

We know already that the dividend and divisor may be multiplied by the same number without affecting the value of the quotient, and it is easy to show that the terms of a fractional number may be so multiplied, though not considered as representing an indicated division.

For instance, multiplying both terms of the fraction $\frac{3}{4}$ by 2 the result is $\frac{6}{8}$.

Now $\frac{3}{4}$ means that a thing is divided into 4 equal parts and that 3 of these parts are reckoned. If each of the fourths were subdivided again into two equal parts each one would obviously be an eighth, and 6 of these would be formed from 3 fourths. In other words, $\frac{3}{4}$ is equivalent to $\frac{6}{8}$.

In general, then, to multiply the denominator is to multiply the number of equal parts, the value of one of which must be so much the smaller. The number taken to represent the same value must be increased accordingly; that is, the numerator must be multiplied by the same number which multiplies the denominator.

If both terms of a fraction may be multiplied by the same factor without changing the value, it obviously follows that the terms may be divided by the same number without changing the value, since the result thus obtained would reproduce the original by multiplying its terms by the number used as a divisor.

Thus dividing both terms of the fraction $\frac{12}{21}$ by 3, the result is $\frac{4}{7}$. Now if both terms of $\frac{4}{7}$ be multiplied by 3 the result will be $\frac{12}{21}$; therefore $\frac{4}{7} = \frac{12}{21}$, or $\frac{12}{21} = \frac{4}{7}$.

A fraction is said to be reduced to its lowest terms when all the factors common to numerator and denominator are divided or cancelled out, or what amounts to the same, when each is divided by the highest common factor. Thus $\frac{18}{27}$ may be reduced to its lowest terms by dividing numerator and denominator by 9, giving the result $\frac{2}{3}$.

Second Reduction.

To change an improper fraction to the form of a whole or mixed number.

If the numerator of an improper fraction be divided by the denominator the quotient will be a whole or mixed number. This quotient will evidently express the true value of the fraction when regarded as a symbol of division. It will also express the true value of the fraction when regarded as indicating a number of equal parts, since the denominator shows how many of those parts are required to make the value of one thing.

For instance, suppose a man has 10 quarters or fourths of a dollar, he would have the value of $2\frac{2}{4}$ dollars, that is, the quotient of 10 divided by 2. In any other like case the reasoning would be similar.

In general, then, *to change an improper fraction to the form of a whole or mixed number, divide the numerator by the denominator.*

Third Reduction.

To change a whole number or a mixed number to the fractional form.

In the case of a whole number multiply by the required denominator, and the product will evidently be the numerator of the fraction sought.

The fraction may then be written.

Suppose it be required to reduce 8 to fourths. Each unit of the 8 will form 4 fourths, and the whole 8 units will evidently form 8 times 4 fourths, that is, 32 fourths, which may be written $\frac{32}{4}$.

In like manner, 11 may be reduced to 11×5 fifths $(= \frac{55}{5})$, or 11×6 sixths $(= \frac{66}{6})$, and so on.

In the case of a mixed number, if the integral part be multiplied by the denominator of the fractional part, the product will express parts of the same kind as the fractional part of the mixed number, and this product may then be added to the numerator of the fractional part, and the sum will be the numerator of the fraction sought, which may now be written.

Thus, to change $8\frac{3}{4}$ to a fractional form: $4 \times 8 = 32$, that is, 8 units are equivalent to 32 fourths, which with 3 fourths given in the fractional part, make 35 fourths in all, and may be written $\frac{35}{4}$.

In a similar way, $9\frac{2}{3} = \frac{29}{3}$, because $3 \times 9 + 2 = 29$, and writing the figure 3 under the figures 29, we have $\frac{29}{3}$.

Then, to change any mixed number to a fractional form, *multiply the whole number by the denominator, and to the product add the numerator: the sum will be the numerator, and the given denominator will be the denominator of the fraction sought.*

Fourth Reduction.

To reduce two or more fractions to similar form.

Fractions are said to be similar when the denominators are equal.

Thus, $\frac{3}{4}$, $\frac{5}{4}$, $\frac{11}{4}$, are similar fractions, so are $\frac{2}{13}$, $\frac{8}{13}$, $\frac{19}{13}$, but $\frac{3}{4}$, $\frac{7}{8}$, are not similar.

It is often desirable to reduce such fractions to similar forms, or, as it is sometimes described, to reduce fractions to a common denominator.

Since the two terms of any fraction may be multiplied by the same factor without affecting the value, it is evident, in the case of two dissimilar fractions, that if both terms of each are multiplied by the denominator of the other fraction, there will result two similar fractions. For example, consider $\frac{3}{4}$, $\frac{7}{8}$. Multiplying both terms of the first by 8 (the denominator of the second), the result is $\frac{24}{32}$. Multiplying both terms of the second fraction by 4 (the denominator of the first), the result is $\frac{28}{32}$, and the two fractions $\frac{3}{4}$, $\frac{7}{8}$, are changed respectively to $\frac{24}{32}$, $\frac{28}{32}$, which are similar, or have equal denominators.

In case of three or more fractions to be reduced, it is evident that if the terms of each fraction be multiplied by the product of all the other denominators, the fractions thus obtained will be similar.

For example, $\frac{2}{3}$, $\frac{3}{4}$, $\frac{5}{6}$, may be changed in this manner to $\frac{48}{72}$, $\frac{54}{72}$, $\frac{60}{72}$ respectively—multiplying the terms of the first fraction by 4×6, those of the second by 3×6, those of the third by 3×4.

It often happens, however, that some smaller number may be found for a common denominator.

In the case of the fractions just considered, that is, $\frac{2}{3}$, $\frac{3}{4}$, $\frac{5}{6}$, by multiplying the terms of $\frac{2}{3}$ by 4, the result is $\frac{8}{12}$; multiplying the terms of $\frac{3}{4}$ by 3, the result is $\frac{9}{12}$; and multiplying the terms of $\frac{5}{6}$ by 2, the result is $\frac{10}{12}$, and all the fractions thus obtained have the common denominator 12.

It is evident that any common multiple of the denominators may serve as the common denominator, and therefore the least common multiple of all

the denominators will be the least common denominator.

It is evident, then, to reduce several fractions to other equivalent fractions having the least common denominator, we may first find the least common multiple of all the denominators, then divide this multiple by the denominator of any fraction, and multiply both terms of that fraction by this quotient.

Apply this operation to each of the fractions.

For example, to reduce the fractions $\frac{1}{2}$, $\frac{2}{3}$, $\frac{1}{4}$, $\frac{1}{8}$ to fractions with the least common denominator.

The least common multiple of the denominators 2, 3, 4, and 8, is found, by the method explained in Chapter VIII., to be 24.

Dividing 24 by 2, the denominator of the fraction $\frac{1}{2}$, the quotient is 12. Multiplying both terms of this fraction by 12, the result is $\frac{12}{24}$. Treating the fraction $\frac{2}{3}$ in a similar way, we find the quotient of 24 divided by 3, to be 8, and multiplying both terms by 8, the result is $\frac{16}{24}$. In a similar way, from $\frac{1}{4}$ is derived $\frac{6}{24}$, and from $\frac{1}{8}$, $\frac{3}{24}$.

Thus the fractions $\frac{1}{2}$, $\frac{2}{3}$, $\frac{1}{4}$, $\frac{1}{8}$, are changed respectively to $\frac{12}{24}$, $\frac{16}{24}$, $\frac{6}{24}$, $\frac{3}{24}$, each having the common denominator 24, which is also the least.

EXERCISES.

(1). Reduce $\frac{24}{18}$ to its lowest terms. *Ans.* $\frac{4}{3}$.

(2). Reduce $\frac{24}{36}$ to its lowest terms. " $\frac{2}{3}$.

(3). Reduce $\frac{36}{42}$ to its simplest form. " $\frac{6}{7}$.

(4). Simplify $\frac{65}{85}$. " $\frac{13}{17}$.

(5). Simplify $\frac{96}{112}$. " $\frac{6}{7}$.

(6). Simplify $\frac{147}{189}$. *Ans.* $\frac{7}{9}$.
(7). Simplify $\frac{4095}{4375}$. " $\frac{117}{125}$.
(8). Simplify $\frac{221}{247}$. " $\frac{17}{19}$.
(9). Simplify $\frac{391}{136}$. " $\frac{23}{8}$.
(10). Simplify $\frac{2079}{1232}$. " $\frac{27}{16}$.
(11). Find the whole number equal in value to $\frac{256}{8}$.
Ans. 32.
(12). Change $\frac{66}{11}$ to an equivalent whole number.
Ans. 6.
(13). Change $\frac{1350}{25}$ to an equivalent whole number.
Ans. 54.
(14). Change $\frac{37}{4}$ to the form of a mixed number.
Ans. $9\frac{1}{4}$.
(15). Change $\frac{72}{19}$ to the form of a mixed number.
Ans. $3\frac{15}{19}$.
(16). Change each of the fractions in the following list to the form of a whole or mixed number: $\frac{81}{10}$, $\frac{343}{7}$, $\frac{34976}{288}$, $\frac{1333}{222}$, $\frac{4444}{55}$.
(17). Reduce $4\frac{3}{4}$ to the form of an improper fraction.
Ans. $\frac{19}{4}$.
(18). Reduce $3\frac{15}{19}$ to a fractional form. *Ans.* $\frac{72}{19}$.
(19). Reduce 6 to the fractional form of 11ths.
Ans. $\frac{66}{11}$.
(20). Reduce $8\frac{1}{10}$ to a fractional form. " $\frac{81}{10}$.
(21). Reduce 12 to 5ths. " $\frac{60}{5}$.
(22). Reduce 18 to 18ths. " $\frac{324}{18}$.
(23). Reduce $7\frac{1}{2}$ to 4ths. " $\frac{30}{4}$.
(24). Reduce $\frac{2}{3}$, $\frac{3}{4}$, $\frac{4}{5}$, $\frac{5}{6}$ to fractions having the least common denominator. *Ans.* $\frac{40}{60}$, $\frac{45}{60}$, $\frac{48}{60}$, $\frac{50}{60}$.
(25). Reduce $\frac{1}{2}$, $\frac{1}{4}$, $\frac{1}{8}$, $\frac{1}{16}$ to fractions having the least common denominator. *Ans.* $\frac{8}{16}$, $\frac{4}{16}$, $\frac{2}{16}$, $\frac{1}{16}$.

(26). Reduce $\frac{4}{15}$, $\frac{3}{25}$, $\frac{5}{7}$ to fractions of similar form.

Ans. $\frac{140}{525}$, $\frac{63}{525}$, $\frac{375}{525}$.

(27). Reduce $\frac{1}{15}$, $\frac{1}{17}$ to similar fractions.

Ans. $\frac{17}{255}$, $\frac{15}{255}$.

(28). Reduce $\frac{1}{10}$, $\frac{1}{11}$, $\frac{1}{12}$ to fractions with the least common denominator. *Ans.*

(29). Reduce 8, $5\frac{1}{2}$, $\frac{9}{10}$ to fractions having the least common denominator. *Ans.*

(30). Reduce $\frac{12}{9}$, $\frac{9}{12}$, $\frac{7}{8}$, $\frac{8}{7}$ to fractions with the least common denominator. *Ans.*

REVIEW X.

a. **The operations of addition, subtraction, multiplication, and division** are applicable to fractional as well as to integral numbers, but changes in the form of fractions are often required, in order that these operations may be conveniently applied.

b. **Any change in the form** of a fraction that does not affect its value, is called a reduction of the fraction.

c. **The reduction** most frequently required consists in multiplying or dividing both terms of a fraction by the same factor.

This operation obviously does not affect the

value of the fraction, regarded as an indication of division, neither when regarded as expressing a number of equal parts, because if each of the equal parts of a unit be increased or diminished, the number of them used to express the same value, must be increased or diminished; in other words, if the denominator be multiplied or divided, the numerator must be multiplied or divided to express the same value.

d. **A fraction** is reduced to its lowest terms by dividing both terms by their greatest common factor.

e. **A second reduction of fractions** frequently used consists in changing an improper fraction to the form of a whole or mixed number, which is done by dividing the numerator by the denominator.

f. **A third reduction of fractions** consists in changing a whole or mixed number to the form of an improper fraction.

This is done by multiplying the whole number by the denominator, adding the numera-

tor (if any) to the product, then writing this sum as a numerator, over the given denominator.

g. **A fourth reduction** consists in reducing two or more fractions to similar forms, that is, to fractions having a common denominator. Usually the least common denominator is required.

It is accomplished by finding the least common multiple of the given denominator, then dividing this multiple by the denominator of each fraction, and multiplying each quotient by the numerator of the same fraction from which the quotient was derived.

Each product thus obtained will be a numerator, and the least common multiple will be the least common denominator required.

CHAPTER XI.

Addition and Subtraction of Fractional Numbers.

If one should have pieces of money, some half dollars and some quarter dollars, and should count them over to find the value of the whole, he could not reckon all the pieces as half dollars, nor all as quarter dollars. He might, however, reckon each half dollar as equal in value to two quarter dollars, or the whole number of half dollars as equivalent to two times as many quarters. Then adding the number of quarters to two times the number of halves, the sum would be the value of the whole reckoned in quarters.

Suppose, for instance, one had 5 half dollars and 7 quarter dollars. The 5 half dollars would be equal in value to 10 quarters, which, added to 7 quarters, would give 17 quarters as the value of the whole, written $\frac{17}{4}$, or $4\frac{1}{4}$ dollars.

The method just used is essentially the same as is used in the addition of any two fractional numbers.

Only things of the same kind can be counted together; that is, in the case of fractional numbers, they should be all halves, or all fourths, or all fifths, or all twentieths, or all of whatever the kind happened to be.

If not at first of the same kind, they should be reduced to the same kind; that is, be reduced to a common

denominator, and then the sum of the numerators will be the numerator, and the common denominator will be the denominator of the sum required.

Thus to add $\frac{1}{2}$ and $\frac{2}{3}$, first they are reduced to a common denominator and become $\frac{3}{6}$ and $\frac{4}{6}$, which, being counted or added together, give the sum $\frac{7}{6} = 1\frac{1}{6}$.

The operation may be expressed as follows:

$$\tfrac{1}{2}+\tfrac{2}{3} = \tfrac{3}{6}+\tfrac{4}{6} = \tfrac{7}{6} = 1\tfrac{1}{6}.$$

In a similar way to add $\frac{1}{4}$, $\frac{3}{5}$, and $\frac{7}{8}$, we may write:

$$\tfrac{1}{4}+\tfrac{3}{5}+\tfrac{7}{8} = \tfrac{10}{40}+\tfrac{24}{40}+\tfrac{35}{40} = \tfrac{69}{40} = 1\tfrac{29}{40}.$$

In the subtraction of fractional numbers there is no further difficulty. As things subtracted must be of the same kind, the fractional numbers must have the same denominator or be reduced to the same denominator. Then the difference of the numerators will be the numerator, and the common denominator will be the denominator of the difference of the two given fractional numbers.

Thus the following operation of subtracting $\frac{5}{7}$ from $\frac{3}{4}$ will be readily understood:

$$\tfrac{3}{4} - \tfrac{5}{7} = \tfrac{21}{28} - \tfrac{20}{28} = \tfrac{1}{28}, \textit{ ans.}$$

Or again to subtract $\frac{3}{10}$ from $\frac{5}{8}$:

$$\tfrac{5}{8} - \tfrac{3}{10} = \tfrac{25}{40} - \tfrac{12}{40} = \tfrac{13}{40}, \textit{ ans.}$$

If, in the case of addition, the numbers are mixed, the whole numbers and the fractional may be added separately, counting the two sums together.

Thus, to add $4\frac{1}{2}$, $2\frac{1}{3}$, $3\frac{3}{4}$; $4+2+3 = 9$, first partial sum. $\frac{1}{2}+\frac{1}{3}+\frac{3}{4} = \frac{6}{12}+\frac{4}{12}+\frac{9}{12} = \frac{19}{12} = 1\frac{7}{12}$, second partial sum. $9+1\frac{7}{12} = 10\frac{7}{12}$, *ans.*

So in the subtraction of mixed numbers, the whole

and fractional parts may be subtracted separately, uniting the results.

Thus to subtract $2\frac{1}{3}$ from $4\frac{1}{2}$. $4 - 2 = 2$, and $\frac{1}{2} - \frac{1}{3} = \frac{3}{6} - \frac{2}{6} = \frac{1}{6}$; or uniting the two results, $2\frac{1}{6}$ is the difference sought.

It is also obvious that the mixed numbers may be reduced to fractional numbers, and then either be added or subtracted, as may be required.

The last example may be operated as follows:

$$4\tfrac{1}{2} = \tfrac{9}{2},\ 2\tfrac{1}{3} = \tfrac{7}{3}.$$

$\frac{9}{2} - \frac{7}{3} = \frac{27}{6} - \frac{14}{6} = \frac{13}{6} = 1\frac{1}{6}$, as before. In all cases reduce the result to the simplest form, if not already so.

It may also be noticed that, regarding fractions as indicated divisions, the operation of addition or subtraction would be the same, since the sum or difference of two quotients is the same as the quotient of the sum or difference of the two dividends, the divisor being the same throughout.

EXERCISES.

(1). Add $\frac{2}{5}$, $\frac{3}{8}$, $\frac{1}{12}$. *Ans.* $\frac{103}{120}$.

(2). Add $\frac{3}{4}$, $\frac{19}{24}$, $\frac{3}{16}$. *Ans.* $\frac{83}{48} = 1\frac{35}{48}$.

(3). Prove that $2\frac{1}{8} + 3\frac{1}{6} + \frac{1}{4} = 5\frac{13}{24}$.

(4). Prove that $\frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} = 1\frac{9}{20}$.

(5). Prove $\frac{5}{6} - \frac{3}{4} = \frac{1}{12}$.

(6). Prove $6\frac{9}{10} - 3\frac{7}{8} = 3\frac{1}{40}$.

(7). Prove $\frac{8}{9} - \frac{11}{21} = \frac{23}{63}$.

(8). Add $317\frac{2}{5}$, $17\frac{3}{51}$, $4\frac{9}{10}$, $\frac{7}{15}$. *Ans.* $339\frac{421}{510}$.

(9). Subtract $2\frac{37}{39}$ from $4\frac{7}{11}$. "

(10). Add $4\frac{7}{15}$, $8\frac{5}{21}$, $2\frac{8}{35}$. "

(11). From $25\frac{7}{10}$ take $24\frac{3}{4}$. "

(12). From $\frac{9}{35}$ take $\frac{7}{30}$. "

REVIEW XI.

a. **In order to add or subtract fractional numbers,** they must have the same denominator, or be reduced to a common denominator, because only things of the same kind can be added or subtracted, and the denominator expresses the kind of thing—as halves, thirds, fourths, and so on.

b. Then to add or subtract fractional numbers with a common denominator, add or subtract the numerators and use the result for the numerator, and the common denominator as the denominator of the fractional sum or difference sought.

c. The same method evidently results when fractions are regarded as indicated divisions.

d. **Mixed numbers** may be reduced to the fractional form, or the integral parts and the fractional parts may be operated on separately (combining the results), as may be found most convenient.

e. In all cases the result should be expressed in the simplest form; that is, the fraction should be reduced to its lowest terms.

CHAPTER XII.

Multiplication of Fractional Numbers.

MULTIPLICATION is essentially of the same nature, whether fractional or integral numbers are used.

There are two cases which may be considered separately. First, when the multiplicand is fractional; second, when the multiplier is fractional. The first offers but little difficulty.

In multiplying the numerator, the denominator remaining the same, it is evident a fractional number is multiplied, because the numerator expresses the number of parts. Thus $\frac{3}{4} \times 2 = \frac{6}{4}$.

Also, to divide the denominator obviously has the effect of multiplying a fractional number, since the value of each of the equal parts is multiplied so many times while the number of the parts remains the same.

For instance, dividing the denominator of $\frac{7}{12}$ by 4 the result is $\frac{7}{3}$. Each of 3 equal parts of a thing is evidently equal to 4 times one of 12 equal parts of the same thing, and there being 7 of the parts in each case it follows that $\frac{7}{3} = \frac{7}{12} \times 4$.

Hence we may say, *to multiply a fractional number by a whole number, multiply the numerator, or divide the denominator, as seems most convenient.*

In considering the case of a fractional multiplier let

us recur to the first meaning of multiplication, which is the counting of a number of things a number of times together.

The operation of counting the multiplicand one time, or multiplying by 1, reproduces the multiplicand. What, then, is the effect of multiplying by $\frac{1}{2}$? Obviously to give half the operation of counting the multiplicand once. If two halves of the operation give the multiplicand, one-half the operation will give half the multiplicand.

To multiply by $\frac{1}{2}$, then, gives a half of the multiplicand, the same as dividing by 2. To multiply by $\frac{1}{3}$ gives a third of the multiplicand, or the same as dividing by 3.

To multiply by $\frac{2}{3}$ is to make two-thirds of an operation of counting once; that is, to give two times one-third.

Or, in case of any fractional multiplier, such a portion of the multiplicand is counted as is indicated by the denominator, and this portion is counted as many times as are indicated by the numerator of the multiplier. That is, the multiplicand is divided by the denominator, and this quotient is multiplied by the numerator.

Of course the result is the same if we first multiply by the numerator and then divide this product by the denominator.

To multiply 12 by $\frac{2}{3}$ the operation would be: $12 \div 3 = 4$, and $4 \times 2 = 8$, or $12 \times 2 = 24$, $24 \div 3 = 8$.

Then, in any case of a fractional multiplier, we may multiply by the numerator and divide by the denomi-

nator, and this is obviously true whether the multiplicand be a whole number or a fractional one.

We have seen that dividing the denominator has the effect of multiplying a fractional number, and it may be easily shown that multiplying the denominator has the effect of dividing a fractional number; for whatever increases the number of parts of a thing diminishes the value of each one, and multiplying the number of parts would therefore divide the value of each.

It often happens, in dividing the numerator of the multiplicand by the denominator of the multiplier, that the dividend is not exact. In that case it is more convenient to multiply the denominator.

For instance, to multiply $\frac{5}{6}$ by $\frac{7}{9}$. First multiplying by the numerator 7 we have $\frac{5}{6} \times 7 = \frac{35}{6}$, and then, as the numerator is not exactly divisible by the denominator 9, we multiply it into the denominator 6, as follows:

$$\frac{35}{6 \times 9} = \frac{35}{54}, \text{ or } \frac{5}{6} \times \frac{7}{9} = \frac{5 \times 7}{6 \times 9} = \frac{35}{54}.$$

It seems, in fact, that in multiplying several fractional numbers together, the product of the numerators is the numerator, and the product of the denominators is the denominator of the required product.

The operation indicated in the following example will be easily understood:

$$\frac{1}{2} \times \frac{2}{3} \times \frac{3}{4} \times \frac{4}{5} = \frac{24}{120}.$$

But this operation may be shortened by means of cancelling (explained in Chapter VII.) as follows:

$$\frac{1}{\cancel{2}} \times \frac{\cancel{2}}{\cancel{3}} \times \frac{\cancel{3}}{\cancel{4}} \times \frac{\cancel{4}}{5} = \frac{1}{5}.$$

This is, in fact, equal to $\frac{24}{120}$, previously obtained, and is what that becomes when reduced to its lowest terms.

The operations indicated in the following examples will be easily understood, and will illustrate the foregoing:

$$(1).\quad \overset{3}{\cancel{6}} \times \frac{3}{\underset{2}{\cancel{4}}} \times \frac{5}{\underset{3}{\cancel{9}}} \times \frac{\overset{3}{\cancel{6}}}{7} \times \frac{1}{\cancel{2}} = \frac{15}{14}.$$

$$(2).\quad \frac{2}{3} \times \frac{5}{7} \times \frac{8}{13} = \frac{80}{273}.$$

$$(3).\quad \frac{4}{5} \times 6 \times \frac{5}{8} \times \frac{4}{7} \times \frac{7}{3} = 4.$$

$$(4).\quad \frac{3}{4} \times 12 = 9.$$

$$(5).\quad 8 \times \frac{6}{5} = \frac{48}{5}.$$

$$(6).\quad 9 \times \frac{4}{3} = 12.$$

Fractions are sometimes connected by the word *of* placed between them, as $\frac{2}{3}$ of $\frac{4}{5}$ (read two-thirds of four-fifths), or $\frac{5}{6}$ of $\frac{6}{4}$ of $\frac{8}{9}$. Such expressions are called compound fractions.

In this case the word *of* is equivalent to the symbol of multiplication, and a compound fraction may be reduced by multiplying together the numerators for a numerator, and multiplying the denominators together for a denominator. Thus

$$\tfrac{5}{6} \text{ of } \tfrac{6}{4} \text{ of } \tfrac{8}{9} = \tfrac{10}{9}.$$

If multiplier or multiplicand happens to be a mixed number, it is usually more convenient to reduce this to a fractional form. Thus

$$4\tfrac{1}{2} \times \tfrac{5}{6} = \tfrac{9}{2} \times \tfrac{5}{6} = \tfrac{15}{4}.$$

Before leaving this subject it is well to notice changes in the forms of some numbers that facilitate the process of multiplication.

Thus $3\frac{1}{3} = \frac{10}{3}$, and in multiplying any whole number by $3\frac{1}{3}$ it is only necessary to add a cipher to the figures of the multiplicand and divide the number thus expressed by 3. For example—

$$\begin{array}{r} 3)288 \\ 3\frac{1}{3} \\ \hline 960 \end{array}$$

Again, $25 = \frac{100}{4}$. Hence $195 \times 25 = \frac{19500}{4} = 4875$.

The application of the relations expressed in the following list will be easily understood:

$$33\frac{1}{3} = \frac{100}{3}, \quad 2\frac{1}{2} = \frac{10}{4}, \quad \frac{5}{6} = \frac{10}{12},$$
$$5 = \frac{10}{2}, \quad 125 = \frac{1000}{8}, \quad 12\frac{1}{2} = \frac{100}{8},$$
$$16\frac{2}{3} = \frac{100}{6}, \quad 6\frac{1}{4} = \frac{100}{16} = \frac{100}{4 \times 4}.$$

In this last case it would be necessary to divide twice by 4. Thus—

$$\begin{array}{r} 4)364 \\ 6\frac{1}{4} \\ \hline 4)9100 \\ \hline 2275 \end{array}$$

By the ordinary method it would be necessary to multiply by 6, divide by 4, and add the two results.

So we have $9 = 10 - 1$, and to multiply by 9 it is sometimes easier to multiply by 10 and subtract the multiplicand.

EXERCISES.

(1). Multiply $\frac{3}{4}$ by $\frac{8}{9}$.	*Ans.*	$\frac{2}{3}$.
(2). Find the product of $\frac{5}{6} \times \frac{9}{15} \times \frac{2}{7}$.	"	$\frac{1}{7}$.
(3). Find the product of $\frac{21}{91} \times \frac{26}{77} \times \frac{28}{6}$.	"	$\frac{4}{11}$.
(4). Find the product of $\frac{1}{2} \times \frac{3}{5} \times \frac{7}{11} \times \frac{13}{17}$.	"	$\frac{273}{1870}$.
(5). Find the product of $4\frac{1}{2} \times \frac{9}{10} \times 40\frac{1}{2}$.	"	$164\frac{1}{40}$.
(6). Multiply $\frac{2}{3}$ of $\frac{3}{4}$ by $\frac{2}{7}$ of 14.	"	2.
(7). Multiply $5\frac{1}{2}$ by $5\frac{1}{2}$.	"	$30\frac{1}{4}$.
(8). Multiply $6\frac{1}{4}$ by $6\frac{1}{4}$.	"	$39\frac{1}{16}$.
(9). Multiply $2\frac{1}{2}$ by $2\frac{1}{2}$.	"	$6\frac{1}{4}$.
(10). Multiply $\frac{1}{2}$ of $\frac{1}{10}$ of $\frac{1}{15}$ by $\frac{4}{3}$ of $37\frac{1}{2}$.	"	$\frac{1}{6}$.
(11). Multiply $\frac{8}{17}$ by 34.	"	16.
(12). Multiply $\frac{21}{38}$ by 95.	"	$52\frac{1}{2}$.
(13). Multiply 38 by $\frac{5}{19}$.	"	10.
(14). Multiply 68 by $\frac{25}{17}$.	"	100.
(15). Multiply 4096 by $\frac{1}{7}$.	"	$585\frac{1}{7}$.
(16). Multiply $5\frac{1}{2} \times 9\frac{1}{2}$ by $\frac{4}{3}$ of $\frac{3}{44}$ of $\frac{4}{19}$.	"	1.
(17). Multiply $\frac{1}{8}$ of 2222 by $\frac{4}{3}$ of 3.	"	1111.
(18). Multiply $\frac{8}{9} \times 40\frac{1}{2}$ by $\frac{1}{10} \times 45$.	"	162.
(19). Multiply $\frac{212}{318}$ by $\frac{999}{555}$.	"	$1\frac{1}{5}$.
(20). Multiply $\frac{375}{500}$ by $\frac{88}{99}$.	"	$\frac{2}{3}$.

(21). Multiply 2524 by 125.

(22). Multiply 1024 by $3\frac{1}{3}$.

(23). Multiply 64 by 25, by $33\frac{1}{3}$, by $8\frac{5}{6}$, by $12\frac{1}{2}$, by 125, by $2\frac{1}{2}$.

REVIEW XII.

a. **Multiplication of fractional numbers** is essentially of the same nature as that of the integral numbers.

b. **A fractional number** is multiplied by multiplying the numerator, the denominator remaining the same; or by dividing the denominator, the numerator remaining the same. Because, in the first case, the number of parts is multiplied, the value of each part remaining the same, while in the second case the value of each part is in effect multiplied, the number remaining the same.

c. Hence, in general, to multiply a fractional number by a whole number, multiply the numerator or divide the denominator, as seems most convenient.

d. **To multiply** is to count the multiplicand some number of times. To multiply by 1 is to count the multiplicand once. To multiply by $\frac{1}{2}$ is to count half the multiplicand, or divide it by 2. To multiply by $\frac{1}{3}$ is to count a third of the multiplicand, or divide it by 3. To multi-

ply by $\frac{2}{3}$ would give 2 times $\frac{1}{3}$, or two times the quotient of the multiplicand divided by 3. And in general to multiply by a fractional number is in effect to divide by the denominator and multiply by the numerator.

To divide by the denominator, either divide the numerator or multiply the denominator, which has the effect of dividing the value of each of the equal parts.

The result, obviously, is the same if we first multiply by the numerator and then divide this product by the denominator.

e. Hence, in any case of a fractional multiplier, we may multiply by the numerator and divide by the denominator. And to divide by the denominator, either divide the numerator or multiply the denominator of the multiplicand, which has the effect of dividing the value of each of the equal parts.

f. In multiplying two or more fractional numbers together (according to the foregoing), the product of the numerators becomes the numerator, and the product of the denominators

becomes the denominator of the required product.

g. But this operation may often be shortened by cancelling any factors common to any numerator and any denominator.

h. **Fractions** are often connected by the word "of" placed between them, which in this case is equivalent to the symbol of multiplication. Fractions so connected are called compound.

i. **Compound fractions** are reduced by multiplying together the fractional numbers represented by them.

j. If either multiplicand or multiplier happens to be a mixed number it is usually more convenient to reduce this to a fractional form at first.

CHAPTER XIII.

Division of Fractional Numbers.

We have already found that dividing the numerator, or multiplying the denominator, has the effect of dividing a fractional number, because, in the first case, the number of the equal parts is divided, the value of each remaining the same; and in the second case the value of each part is divided, the number remaining the same.

Hence, *to divide a fractional number by a whole number, divide the numerator or multiply the denominator, as seems most convenient.* Accordingly, to divide $\frac{12}{17}$ by 4, the quotient is $\frac{3}{17}$.

Or, to divide $\frac{7}{12}$ by 4, the quotient is $\frac{7}{12\times4} = \frac{7}{48}$.

In this case it is more convenient to multiply the denominator than to divide the numerator. Also in the following example: To divide $\frac{4}{15}$ by 5, $\frac{4}{15\times5} = \frac{4}{75}$.

If the divisor is a fractional number, it is less than the numerator, as many times as the denominator. Hence the quotient will be greater than the quotient given by using the numerator alone, as many times as the denominator.

Hence, when the divisor is a fractional number, divide by the numerator, and multiply this quotient by the denominator.

To divide 12 by $\frac{3}{4}$ we have $12 \div 3 = 4$ and $4 \times 4 = 16$.

To divide $\frac{9}{10}$ by $\frac{7}{8}$.

$$\frac{9}{10} \div 7 = \frac{9}{70}, \frac{9}{70} \times 8 = \frac{72}{70} = \frac{36}{35} = 1\frac{1}{35}.$$

A careful examination of this process will make it appear that it is equivalent to interchanging the terms of the divisor and then proceeding as in multiplication.

To interchange the terms of the divisor is to use the numerator as a denominator, and use the denominator as a numerator.

Then, to divide $\frac{9}{10}$ by $\frac{7}{8}$, the process may be more briefly indicated as follows:

$$\frac{9}{10} \times \frac{8}{7} = \frac{72}{70} = \frac{36}{35} = 1\frac{1}{35}.$$

Or again, cancelling out the common factor 2, as follows:

$$\frac{9}{\cancel{10}_5} \times \frac{\cancel{8}^4}{7} = \frac{36}{35} = 1\frac{1}{35}.$$

The solution of the following examples will now be readily understood.

To divide $\frac{3}{5}$ by $\frac{4}{3}$. $\frac{3}{5} \times \frac{3}{4} = \frac{9}{20}$, *ans.*
To divide $\frac{4}{5}$ by $\frac{10}{12}$. $\frac{4}{5} \times \frac{12}{10} = \frac{24}{25}$, "
To divide 8 by $\frac{4}{5}$. $8 \times \frac{5}{4} = 10$, "
To divide $\frac{4}{5}$ by 8. $\frac{4}{5} \times \frac{1}{8} = \frac{1}{10}$, "

It may be remarked that any whole number may be written in the fractional form by giving it the denominator 1, when it may be treated as any other fractional number. Thus 8 may be written $\frac{8}{1}$, and when the terms are interchanged it becomes $\frac{1}{8}$.

The quotient of 1 divided by any number, is called the reciprocal of that number; $\frac{1}{8}$ is the reciprocal of 8, $\frac{1}{3}$ is the reciprocal of 3, and $\frac{1}{10}$ is the reciprocal of 10.

Again $1 \div \frac{1}{8}$ is the reciprocal of $\frac{1}{8}$. That is, $1 \div \frac{1}{8} = 1 \times \frac{8}{1} = 8$, and 8 is the reciprocal of $\frac{1}{8}$. In like manner 3 is the reciprocal of $\frac{1}{3}$, and 10 is the reciprocal of $\frac{1}{10}$.

So $1 \div \frac{3}{5} = \frac{5}{3}$ is the reciprocal of $\frac{3}{5}$, and in general, if the terms of a fractional number be interchanged, the result will be the reciprocal of the fractional number.

Accordingly $\frac{4}{3}$ is the reciprocal of $\frac{3}{4}$, $\frac{7}{8}$ is the reciprocal of $\frac{8}{7}$, and so on.

The methods used in the division of fractional numbers are sufficient to enable us to simplify complex fractions.

But first it may be remarked that a mixed number should, in such cases, be changed to a fractional form.

Thus $\frac{4\frac{1}{2}}{\frac{2}{3}}$, which is read $4\frac{1}{2}$ divided by $\frac{2}{3}$, may be changed to $\frac{\frac{9}{2}}{\frac{2}{3}}$. Performing the operations indicated, that is, dividing $\frac{9}{2}$ by $\frac{2}{3}$, we have $\frac{9}{2} \times \frac{3}{2} = \frac{27}{4} = 6\frac{3}{4}$.

Again, $\frac{\frac{9}{11}}{\frac{3}{4}} = \frac{9}{11} \times \frac{4}{3} = \frac{12}{11} = 1\frac{1}{11}$.

EXERCISES.

(1). Divide $\frac{8}{9}$ by 4. *Ans.* $\frac{2}{9}$.
(2). Divide $\frac{8}{9}$ by 3. " $\frac{8}{27}$.
(3). Divide $\frac{1}{3}$ by 10. " $\frac{1}{30}$.
(4). Divide $2\frac{1}{2}$ by 5. " $\frac{1}{2}$.
(5). Divide $\frac{1}{17}$ by 17. " $\frac{1}{289}$.
(6). Divide $7\frac{3}{10}$ by 18. "
(7). Divide 4 by $\frac{8}{9}$. " $4\frac{1}{2}$.
(8). Divide 3 by $\frac{8}{9}$. " $3\frac{3}{8}$.
(9). Divide 10 by $\frac{1}{3}$. "
(10). Divide 5 by $2\frac{1}{2}$. "

(11). Divide $\frac{231}{484}$ by $\frac{77}{121}$.	*Ans.*	
(12). Divide $48\frac{1}{2}$ by $\frac{194}{306}$.	"	
(13). Simplify $\frac{18\frac{7}{10}}{12\frac{1}{3}}$	"	
(14). Simplify $\frac{88}{88\frac{1}{3}}$.	"	
(15). Divide $\frac{19}{280}$ by $\frac{19}{281}$.	"	
(16). Divide $\frac{414}{3764}$ by $\frac{1890}{2100}$.	"	$\frac{115}{941}$.
(17). Divide $\frac{3322}{2472}$ by $\frac{66}{96}$.	"	$1\frac{1}{2}$.
(18). Divide $\frac{1234}{5678}$ by $\frac{234}{678}$.	"	
(19). Divide $\frac{4448}{5555}$ by $\frac{7272}{25}$.	"	
(20). Find the reciprocal of $4\frac{1}{2}$.	"	$\frac{2}{9}$.
(21). Find the reciprocal of $\frac{1}{2}\times\frac{2}{3}$.	"	3.
(22). Find the reciprocal of $\frac{1}{4}\times 20$.	"	
(23). Find the reciprocal of $\frac{2\frac{1}{3}}{3\frac{1}{3}}$.	"	

(24). Find the reciprocal of $11\frac{1}{2}$, of $8\frac{1}{4}$, of $\frac{2\frac{1}{3}}{12\frac{1}{3}}$.

REVIEW XIII.

a. **To divide a fractional number by a whole number,** divide the numerator or multiply the denominator, as seems most convenient, because in the first case the number of parts is divided, the value of each remaining the same; and in the second case the value of each part is divided, the number remaining the same.

b. **When the divisor is a fractional number,** we may first divide by the numerator, and

then multiply this quotient by the denominator, because the numerator alone is so many times larger than the real divisor.

This is, in effect, equivalent to interchanging the terms of the divisor and then using this result as a multiplier.

c. **The reciprocal of any number** is the quotient of 1 divided by that number. If the number is fractional, its reciprocal is expressed by interchanging its terms.

d. **A complex fraction** is simplified by performing the indicated division according to the method of the division of fractional numbers.

CHAPTER XIV.

Decimal Fractions.

HOW TO WRITE AND HOW TO READ THEM.

THERE is a class of fractional numbers of special importance on account of the convenient form in which they may be written.

This class includes all those whose denominators are 10, or any power of 10, as 100, 1000, and so on.

These fractional numbers are called *decimal*, which signifies "something relating to ten," and by common usage the term decimal, or decimal number, is understood to mean a decimal fractional number.

The convenience of writing decimal fractions consists in omitting the figures of the denominator, and instead placing a point (called a decimal point) at the left of the figures of the numerator. Accordingly $\frac{1}{10}$ may be written .1, $\frac{25}{100}$ may be written .25, and $\frac{748}{1000}$ as .748.

The figures at the right of the decimal point are called decimal figures. In each case the number of decimal figures must equal the number of ciphers that would be required to express the denominator. When needed for this purpose, ciphers must be placed between the other figures of the numerator and the decimal point.

Thus $\frac{2}{100}$ is written .02, $\frac{5}{1000}$ as .005, and $\frac{25}{10000}$ as .0025, and so on.

It will be noticed that a fractional number of the order of tenths requires one decimal figure to express it, the order of hundredths requires two decimal figures, the order of thousandths requires three decimal figures, and so on. In any decimal expression the order of number indicated by the denominator is easily found out by naming each decimal figure, beginning at the left, calling the first *tenths*, the second *hundredths*, the next *thousandths*, and so on to the last, the name of which corresponds to the order of the denominator.

A decimal fraction is read in the same manner in whichever form it is written; that is, by first naming or reading the numerator, and then naming the order of the denominator.

Thus .7 is read 7 tenths and .013 is read 13 thousandths, just as it would be if written $\frac{13}{1000}$.

Suppose it were required to read the fraction .00334567. To find out the order of the denominator we begin at the left and call the order indicated by each figure, as written here for illustration:

	Tenths.	Hundredths.	Thousandths.	Ten-thousandths.	Hund. thousandths.	Millionths.	Ten-millionths.	Hund. millionths.
.	0	0	3	3	4	5	6	7

In this way the order of the right hand figure is found to be *hundred-millionths*, and the fraction is read 334567 hundred-millionths.

Of course it is not required in ordinary practice to write the names of the orders above the figures, for these will be easily remembered without that trouble.

It will be noticed that the orders of numbers indicated by the places of decimal figures diminish in value towards the right, and increase towards the left, at the same rate as in the notation of whole numbers. That is, the value of the order of any place is ten times that on the right, or one-tenth that on the left.

For this reason fractional numbers, written in the decimal form, may be operated on by the same methods as whole numbers, only taking care to place the decimal point correctly.

And, as in the case of whole numbers, a decimal number is equal to the sum of the numbers expressed by the separate figures as written.

For example, the whole number $125 = 1$ hundred $+$ 2 tens $+ 5$, and the decimal $.125 = 1$ tenth $+ 2$ hundredths $+ 5$ thousandths.

The numerator of a decimal fraction may be a mixed number, and in this way the fraction become complex, the same as in the case of common fractions. Thus

$$.004\tfrac{1}{2} = \tfrac{4\frac{1}{2}}{1000}, \text{ or } 33\tfrac{1}{3} = \tfrac{33\frac{1}{3}}{100}.$$

Mixed numbers, with the fractional part expressed in decimal form, are often used, and are read as in other cases, by connecting the names of the integral and fractional parts by the word *and*.

For example, 4.3 is read *four and three-tenths*, and 15.25 is read *fifteen and twenty-five-hundredths*.

Sometimes, in the case of large numbers, and occasionally in other cases, it is desirable to make a more

marked distinction between the integral and fractional parts of a mixed number, and for this purpose the word *decimal* may be pronounced before naming the fractional part.

For example, 1300.0013 may be read "thirteen hundred, and *decimal*, thirteen ten-thousandths."

EXERCISES.

The numbers named in the following list are to be expressed in figures:

(1). One hundred seventy-seven thousandths. .177
(2). Forty-three and seven tenths. 43.7
(3). Eight hundred and twenty-four thousandths. 800.024
(4). Nine hundred seventeen millionths.
(5). Nine hundred seventeen ten-thousandths.
(6). Nine hundred and seventeen ten-thousandths.
(7). Nine hundred seventeen hundredths.
(8). Nine hundred seventeen tenths.
(9). Nine hundred seventeen thousand.
(10). Twenty-seven hundred-millionths.
(11). Twenty-seven hundred millionths.
(12). Five hundred ten ten-millionths.

The following list of decimals is given for exercise in reading:

(1). .47	(6). .0404
(2). .236	(7). 4.3043
(3). 4.444	(8). 12.1212
(4). 1.100	(9). .66666
(5). .0023	(10). 66.33003

(11).	11.11111	(16).	5.10754
(12).	1.0001	(17).	0.10101
(13).	1250.01250	(18).	6.7379999
(14).	1300.00132	(19).	12.0000909
(15).	17.17170	(20).	333.3300333

It may be remarked that placing a cipher at the right of decimal figures is equivalent to multiplying both numerator and denominator by 10, and therefore does not change the value of the number represented.

REVIEW XIV.

a. **A decimal fraction** is one whose denominator is 10, or some power of 10.

b. **The word decimal** signifies "something relating to ten," and in the case of fractional numbers refers to the character of the denominator.

c. **A decimal fractional number** is often called a **decimal number,** or simply a **decimal.**

d. **A decimal fraction** is usually expressed by omitting the figures of the denominator, and instead placing a point, called a decimal point, at the left of the figures of the numerator.

e. **The figures at the right of the decimal point** are called decimal figures, and the num-

ber of decimal figures must equal the number of ciphers that would be required to express the denominator if written. When necessary for this purpose ciphers may be placed between the other figures of the numerator and the decimal point.

f. **A decimal of the order of tenths** requires one decimal figure, the order of hundredths requires two decimal figures, and so on, the last decimal figure at the right being of the order indicated by the denominator.

Hence, to find the order indicated by the denominator of a decimal fraction, begin with the figure next the decimal point, calling that tenths, the next hundredths, and so on to the last, which will be of the same order as the denominator.

g. **A decimal fraction** is read like any other; that is, by first naming the numerator, then naming the order indicated by the denominator.

h. **Decimals may be operated on** by the same methods as whole numbers, only taking

care to put the decimal point in the right place each time.

i. **In reading a mixed number** expressed decimally, as with any form of mixed number, connect the names of the integral and fractional parts by the word **and**, or when required to make the distinction more marked use the words "**and decimal.**"

j. **A cipher** placed at the right of the figures of a decimal fraction, has the effect of multiplying numerator and denominator, each by ten, and does not change the value expressed by the fraction.

CHAPTER XV.

Addition and Subtraction of Decimals.

IN the addition or in the subtraction of fractional numbers expressed in the common form, it was found necessary to reduce them to a common denominator when not already similar.

In the case of decimal fractions no reduction is required, because it is only necessary to add or subtract decimals of the same order; that is, those which have already a common denominator.

Suppose it be required to add .04 and .125. For the sake of convenience place the figures under each other, so that those of the same order will fall in the same column, then add as in whole numbers. Thus,

$$\begin{array}{r} .125 \\ .04 \\ \hline .165 \end{array}$$

Beginning with the lowest order, thousandths, we find only 5, and this is marked down. There are 2 hundredths and 4 hundredths, which are added and give the sum 6 hundredths, and this is also marked down. In like manner the tenths, of which only 1 is found.

If it happen that the sum of the numbers expressed in any column exceeds 10, the number of tens will be

counted in the result of the next column, and the balance marked below the same column, as in whole numbers, and for the same reason.

Suppose we have .333, .1243, and .78962 to be added. Proceeding as before, we have

$$
\begin{array}{r}
.333 \\
.1243 \\
.78962 \\
\hline
1.24692
\end{array}
$$

Or again, to add 140.737, 19.0019, 2004.1007, and .00033, the following operation will be easily understood:

$$
\begin{array}{r}
140.737 \\
19.0019 \\
2004.1007 \\
.00033 \\
\hline
2163.83993
\end{array}
$$

In the subtraction of decimals the foregoing considerations furnish an explanation of most difficulties to be found. One instance will, however, receive particular notice. That is when the figures of the minuend do not extend so far to the right as those of the subtrahend. For example, to subtract .0066 from 5.58. Writing them in proper order we have

$$
\begin{array}{r}
5.58 \\
.0066 \\
\hline
5.5734
\end{array}
\qquad \text{or} \qquad
\begin{array}{r}
5.5800 \\
.0066 \\
\hline
5.5734
\end{array}
$$

Ciphers may be placed at the right of the figures of the minuend without changing the value, so that it becomes

\$.5800, when the difficulty would be obviated, or the subtraction may be performed, as though the ciphers filled the vacant places.

In general, then, to add or subtract decimals, write the figures so that all those of any one order shall be found in the same column, and proceed as in the case of whole numbers.

EXERCISES.

(1). Add .0185, 34.34, 15.0015, 73.0, and 1000.
Ans. 1122.36.

(2). Add 98.594744, .112277, .374629, .18185, .7365.
Ans. 100.

(3). Add 21.33333, 173.55555, 136.11112, 444.44557, and 224.55443. *Ans.* 1000.

(4). Add twenty-seven hundred-thousandths, twelve ten-thousandths, eighteen hundred seventy-five hundred-thousandths, and ninety-seven thousand nine hundred seventy-eight hundred-thousandths. *Ans.* 1.

(5). Add

$$
\begin{array}{r}
.123456 \\
.002003 \\
.111111 \\
.987654 \\
12.343234 \\
17.333333 \\
\hline
\end{array}
$$

Ans.

(6). From 81.846 subtract 18.648. *Ans.* 63.198.

(7). From 75.75 subtract 2.98645. *Ans.*

(8). From 1 subtract .999999. *Ans.*

(9). From thirty-seven ten-thousandths take fourteen hundred-millionths. *Ans.*

(10). Perform the operations indicated below:

$$.0014 + 127.03001 + 1600 + .000099 - 160 - 0.0077 - .333333.$$

Ans.

(11). From one thousand take one thousandth.

(12). Perform the operations indicated as follows:

$$10.10203 + 18.00018 - 10.10203 + 13.987 - .1800018.$$

Ans.

REVIEW XV.

a. **To add or subtract decimals,** write the figures so that those of the same order of number are found in the same column, and then proceed as in whole numbers.

b. **In the case of subtraction,** if it happen that the decimal figures of the subtrahend extend to more places than those of the minuend, either fill the vacant places at the right of the figures of the minuend with ciphers, or proceed as though they were there.

CHAPTER XVI.

Multiplication of Decimals.

In the multiplication of decimals, as in the case of other fractional numbers, the product of the numerators is the numerator, and the product of the denominators is the denominator of the required product.

The number of ciphers required to express the product of the denominators is evidently equal to the sum of the ciphers of the given denominators. Hence the number of decimal figures required to express the product of two decimals, is equal to the sum of the decimal figures of the multiplicand and multiplier.

For example, to multiply .3 by .05, we have $.3 \times .05 = .015$, because the product of the numerators is 15, and the product of tenths into hundredths is thousandths, and the product sought is therefore 15 thousandths, written .015.

And in general, the figures of the product must contain as many decimal figures as those of both the multiplicand and multiplier.

If the multiplier be 10, or any power of 10, the product may be conveniently obtained by dividing the denominator.

To divide the denominator by 10 has the effect to remove the decimal point one place to the right in

the figures of the multiplicand. To divide by 100 removes it two places to the right, and so on.

Hence, if the multiplier be 10, or any power of 10, remove the decimal point as many places to the right in the figures of the multiplicand as there are ciphers in the figures of the multiplier, and the result will express the product.

In some cases it is necessary to annex ciphers to the figures of the multiplicand in order to have places enough. For example, to multiply .25 by 1000, a cipher is written at the right of the figures .25, making .250 (which does not change the value), and then the decimal point is removed to the right of the three figures, giving the result 250.

EXERCISES.

(1). Multiply 23.91 by .538. *Ans.* 12.86358.
(2). Multiply .25 by .15. *Ans.*
(3). Multiply 1887.341702 by 1.0083. "
(4). Multiply .00125 by .33. "
(5). Multiply 1234.5678 by 712.8888. "
(6). Multiply .0268 by 10.025. "
(7). Multiply 888999.111111 by 444.666. "
(8). Multiply 223344.008 by .00033. "
(9). Multiply .05 by .05. "
(10). Multiply .16 by .25. " .04.
(11). Multiply 400 by .0075. " 3.
(12). Multiply 800 by .00125. " 1.
(13). Multiply 37.5 by $.04\frac{1}{3}$. " 1.625.
(14). Multiply 15.15 by $.33\frac{1}{3}$. " 5.05.
(15). Multiply 2.5 by $.03\frac{1}{3}$. " $.083\frac{1}{3}$.

(16).	Multiply .043 by .020$\frac{1}{7}$.	*Ans.*	.000866$\frac{1}{7}$.
(17).	Multiply 12.048 by .002$\frac{1}{2}$.	"	.03012.
(18).	Multiply 40 by .02$\frac{1}{2}$.	"	1.
(19).	Multiply .027 by 33$\frac{1}{3}$.	"	.9.
(20).	Multiply .000$\frac{1}{2}$ by .00$\frac{1}{8}$.	"	.00000$\frac{1}{16}$.

REVIEW XVI.

a. **The decimal figures of any product must be equal in number to the sum of those of multiplicand and multiplier.**

b. **To express multiplication** by any power of 10, remove the decimal point as many places to the right in the figures of the multiplicand as the number of ciphers in the figures of the multiplier.

CHAPTER XVII.

Division of Decimals.

IF dividend and divisor be multiplied by the same factor, the resulting quotient remains unchanged. By this means the divisor may always be made to become a whole number, and the process of division becomes as simple as that of whole numbers.

Let it be required to divide 3.725 by .25.

First multiplying both dividend and divisor by 100 (which is indicated by removing the decimal point two places to the right in the figures of each), the dividend becomes 372.5 and the divisor becomes 25.

The division may then be carried on as follows:

```
25)372.5(14.9
   25
   ---
   122
   100
   ---
    22.5
    225
    ---
```

Dividing the integral part of the dividend by 25, the quotient 14, with a remainder 22, is obtained. This remainder, together with the fractional part of the dividend, is 22.5, which is the same as 225 tenths, and the quotient of this part will obviously be of the order of tenths.

The quotient figure 9 will then be of the order of tenths, and a decimal point is placed between the figures 4 and 9, and the entire quotient is 14.9.

In any case, then, to divide one decimal by another, multiply both dividend and divisor by the least power of 10 that will cause the divisor to become a whole number, then proceed as in the division of whole numbers, marking as many decimal places in the figures of the quotient as those of the dividend actually used.

As another example, let it be required to divide .71875 by .023. Multiplying both numbers by 1000, the dividend becomes 718.75 and the divisor becomes 23. Then dividing, the operation is as follows:

```
23 ) 718.75 ( 31.25
     69
     ---
      28
      23
      ---
       5.7
       46
       ---
       115
       115
       ---
```

It is obvious that each partial quotient is of the same order as the lowest order that appears in the corresponding partial dividend. Thus the lowest order in the first partial dividend (71) is *tens*, and the first partial quotient (3) is of the order of *tens*. The lowest order in the next partial dividend (28) is units, and the next partial quotient (1) is of the order units.

In a similar way the next partial quotient is of the

order *tenths*, and requires the decimal point before the figure expressing it.

It sometimes happens that the figures of the dividend are not so many as those of the divisor, in which case ciphers may be annexed to the decimal figures of the dividend, without changing its value, and the operation proceed as before. Should it also happen that the dividend is a whole number, then a decimal point should be placed at the right of the figures before adding the ciphers.

For example, to divide 15 by 375. Marking a decimal point at the right of the figures of the dividend, and adding ciphers, we have 15.00. Of course any number of ciphers may be added, but in this case only two are needed.

The operation of division in this example is then completed as follows:

```
375)15.00(.04
    1500
    ----
```

The quotient 4 is of the order hundredths, because the order hundredths is used in the dividend to obtain this quotient.

Again, to divide .25 by 12.5. Multiplying both by 10, the divisor becomes a whole number, and the dividend becomes 2.5, and annexing a cipher to these figures, we have 2.50, and the operation of division is as follows:

```
125)2.50(.02
    250
    ---
```

The quotient is of the order hundredths, because that is the lowest order of the dividend used.

It sometimes happens, however far the process of division may be carried, a remainder continues to be found.

In that case the incomplete division may be indicated in the form of a common fraction, discontinuing the operation with any partial quotient at pleasure.

For example, to divide .2 by 7:

$$\begin{array}{r} 7).20(.028\frac{4}{7} \\ \underline{14} \\ 60 \\ \underline{56} \\ 4 \end{array}$$

Or it may be carried further, as follows:

$$\begin{array}{r} 7).20(.02857\frac{1}{7} \\ \underline{14} \\ 60 \\ \underline{56} \\ 40 \\ \underline{35} \\ 50 \\ \underline{49} \\ 1 \end{array}$$

Or, again, the quotient might have been written $.02\frac{6}{7}$, stopping with the first remainder.

EXERCISES.

(1).	Divide .025 by .0005.	*Ans.* 50.
(2).	Divide 24.8 by 400.	" .062.
(3).	Divide 10.10 by 2.5.	" 4.04.
(4).	Divide 281.743 by 1492.	
	Ans. $.1888\frac{534}{1492} = .1888\frac{267}{746}$.	
(5).	Divide 1000.10 by .025.	*Ans.* 40004.
(6).	Divide 30 by .00012.	"
(7).	Divide .00015 by 40.	"
(8).	Divide 300 by .00015.	"
(9).	Divide 2.472 by .309.	"
(10).	Divide 1024 by .32.	"
(11).	Divide .0404 by 202.	"
(12).	Divide 1300.28 by .125.	"
(13).	Divide 12.874 by $0.33\frac{1}{3}$.	"
(14).	Divide 1.81972 by $.006\frac{1}{4}$.	"
(15).	Divide 247.2 by $5.1\frac{1}{2}$.	"
(16).	Divide $2.4\frac{3}{4}$ by 24.	"
(17).	Divide 2000.0002 by .75.	"
(18).	Divide 1.876 by 4832.4.	"
(19).	Divide .00024 by .016.	"
(20).	Divide .000075 by 2400.	"
(21).	Divide $.33\frac{1}{3}$ by 30.	"
(22).	Divide $14.66\frac{2}{3}$ by 25.	"

REVIEW XVII.

a. **In the division of decimals** both dividend and divisor may be multiplied by the lowest power of 10 that will cause the divisor to become a whole number, and the process of division may then be carried on as in the case of whole numbers. The decimal figures of the quotient will then be as many as those of the dividend actually used.

b. **Ciphers may be annexed to the decimal figures of the dividend** at any stage of the process, or if the dividend be a whole number a decimal point may be marked at the right of the figures, and ciphers then annexed.

c. **If it happens that a remainder is found,** however far the process of division is carried, the process may be stopped at any desired point, the incomplete division being expressed in the form of a common fraction.

d. **Any whole number** may be divided by any other whole number, and the quotient, if not integral, be expressed decimally.

CHAPTER XVIII.

Reduction of Decimals.

As any numerator may be divided by any denominator, and the quotient be expressed decimally, so any fractional number may be expressed in the decimal form, either simple or complex. Simple when the division finally terminates without any remainder, and complex when the quotient is expressed in part by a fraction of the common form.

For example, $\frac{3}{5} = \frac{3.0}{5} = .6$. Again, $\frac{3}{4} = \frac{3.00}{4} = 75$. And $\frac{2}{3} = \frac{2.0}{3} = .6\frac{2}{3}$.

It may be shown that in reducing a fractional number from the common to the decimal form, if the division does not terminate with an exact quotient, the figures of the quotient will finally recur in the same order and continue to repeat themselves indefinitely. Such a decimal is called a *repeating* or *circulating decimal*, and sometimes a *circulate*.

For example, $\frac{2}{3} = .6666\frac{2}{3}$, and however far the division is carried, the quotient figure 6 is obtained each time. Again, $\frac{2}{7} = .28571\dot{4}\dot{2}8571\dot{4}$, etc.

If there is only one repeating decimal figure, that fact is indicated by marking a dot above that figure, as $\frac{2}{3} = .\dot{6}$.

If there are two or more repeating decimal figures,

a dot is marked over the first and the last of the series. Thus $\frac{1}{11} = .0909\dot{0}\dot{9}$ and $\frac{4}{11} = 3636\dot{3}\dot{6}$. Of course the decimal figures may be extended at pleasure, and it makes no difference which of the figures is first marked with a dot, provided the proper number of places intervene before the second dot is marked.

Thus $\frac{2}{11} = .1818\dot{1}\dot{8}$, or $\frac{2}{11} = .181\dot{8}\dot{1}$, or $\frac{2}{11} = .\dot{1}81\dot{8}$.

Again, $\frac{115}{333} = 3.45\dot{3}4\dot{5}$, or it equals $.34534\dot{5}3\dot{4}$, or $= .3453\dot{4}5\dot{3}$.

To reduce a simple or complex decimal fraction to the common form only requires the decimal point to be omitted, and the figures of the denominator to be written underneath those of the numerator in the usual manner. Thus .35 may at once be written $\frac{35}{100}$, or $.013\frac{1}{3}$ may be written $\frac{13\frac{1}{3}}{1000}$, and so on. Of course a complex fraction (like the last one) may be further reduced to the simple form. As $\frac{13\frac{1}{3}}{1000} = \frac{13\frac{1}{3} \times 3}{1000 \times 3} = \frac{40}{3000} = \frac{4}{300} = \frac{1}{75}$.

The reduction of a repeating decimal to the common form is not so obvious, but even this will be found to offer no great difficulty.

Let it be required to reduce $.\dot{7}$ to the common form. $.\dot{7} = .7\dot{7}$, since it makes no difference how far the repeating figures are carried. Multiplying by 10 we may write $10 \times .\dot{7} = 7.\dot{7}$.

If, now, the original circulate $.\dot{7}$ be subtracted from this, the remainder is exactly 7., as may be seen:

$$\begin{array}{r} 7.\dot{7} \\ .\dot{7} \\ \hline 7.0 \end{array}$$

For however far the repeating figures be carried they

will be just the same in the minuend and subtrahend. This difference 7. is the difference between ten times a number and once (or 1 time) the same number. But the difference between once and ten times a number is obviously 9 times the number.

If, then, 7 is 9 times a number, that number must be $\frac{1}{9}$ of 7; that is, $\frac{7}{9}$. And, indeed, it will appear at once that changing the common fraction $\frac{7}{9}$ to the decimal form gives the repeating decimal $.\dot{7}$.

By a process similar to the foregoing $.\dot{8}$ may be reduced to $\frac{8}{9}$. Suppose we consider $.0\dot{8}$ instead of the above, and apply the same method. We have $.0\dot{8} = .08\dot{8}$, and $10 \times .08\dot{8} = .8\dot{8}$. Subtracting $.0\dot{8}$ from this there remains .8, or $\frac{8}{10}$, which must be 9 times the number sought. But $\frac{1}{9} \times \frac{8}{10} = \frac{8}{90}$. Hence $.0\dot{8} = \frac{8}{90}$. This result might have been reached by other considerations, for $.0\dot{8}$ is obviously $\frac{1}{10}$ of $.\dot{8}$; and since $.\dot{8} = \frac{8}{9}$, then $\frac{1}{10}$ of $.\dot{8}$, or $.0\dot{8}$, must $= \frac{1}{10} \times \frac{8}{9} = \frac{8}{90}$.

Again, suppose we have $.\dot{3}\dot{5}$ given for reduction. Here are two figures that repeat. If this be multiplied by 10, as in the previous case, the result is $3.\dot{5}\dot{3}$, and subtracting $.\dot{3}\dot{5}$, the difference would not be free from a repeating part. If, however, we multiply by 100, the product is $35.\dot{3}\dot{5}$, and in subtracting $.\dot{3}\dot{5}$ the difference is found to be exactly 35. But the difference between a number and 100 times the number must be 99 times the number. If 35 be 99 times a number, that number must be $\frac{35}{99}$. Hence $.\dot{3}\dot{5} = \frac{35}{99}$.

In any case of reducing a repeating decimal we may multiply by 10, or some power of 10, so that the difference between this product and the given decimal shall

be a simple decimal without any repeating part. Then dividing this difference by 1 less than the number used as a multiplier, the quotient expressed in the form of a fraction is the result sought.

The examples which follow will illustrate this precept. To reduce $1.23\dot{1}4\dot{3}$ to a common fraction:

$$\begin{aligned} 1000 \times 1.23\dot{1}4\dot{3} &= 1231.43\dot{1}4\dot{3} \\ 1 \times 1.23\dot{1}4\dot{3} &= 1.23\dot{1}4\dot{3} \\ \hline 999 \times 1.23\dot{1}4\dot{3} &= 1230.2 \\ 1.23\dot{1}4\dot{3} &= \tfrac{1230.2}{999} = \tfrac{12302}{9990} = 1\tfrac{1156}{4995}. \end{aligned}$$

To change $.00\dot{2}66\dot{8}$ to the common form:

Multiplying by 10000 the product is $26.68\dot{2}66\dot{8}$, and now subtracting the given decimal, $.00\dot{2}66\dot{8}$, the difference is 26.68.

Dividing by 9999 we have $\frac{26.68}{9999} = \frac{2668}{999900}$. *Ans.*

Again to change $1.0001\dot{1}$ to the common form:

$$\begin{aligned} 10 \times 1.00001\dot{1} &= 10.00011\dot{1} \\ 1.00001\dot{1} &= 1.00001\dot{1} \\ \hline & 9.00010 \end{aligned}$$

$$\frac{9.000010}{9} = \frac{900001}{900000}.$$

To make repeating decimals coterminous.

Repeating decimals are said to be coterminous if in the figures of each there are the same number that repeat, ending with the same order of number.

Thus $.0\dot{2}\dot{3}$ and $.00\dot{2}\dot{3}$ are not coterminous, because the sets do not end with the same order of number, but it is obvious that a slight change will make them coterminous, for the first may be written $.02\dot{3}\dot{2}$, and it is now coterminous with the other.

Again, suppose we compare $.0\dot{2}\dot{3}$ and $.0\dot{2}3\dot{4}$.

The first fraction has two repeating figures, the other has three.

The first may, however, be changed to a six-place repeater by writing as follows: $.0\dot{2}3232\dot{3}$, and the second may be written $.0\dot{2}3423\dot{4}$, and the two become coterminous.

It is evident, then, that any two repeating decimals may be made to have the same number of repeating places by finding the least common multiple of the two numbers of repeating places, and extending the number of repeating places in each decimal to that limit.

Thus if one set has 3 places and another has 4, each one may be extended to 12 places.

When the two sets have the same number of repeating places, but these do not begin with the same order, it is only necessary to transfer the beginning and ending of either set a sufficient number of places to the right, as in one of the previous examples, and as in the following:

To make $8.593\dot{6}7\dot{6}$ and $0.01\dot{2}3\dot{7}$ coterminous.

It is only necessary to transfer the set of three places in the second decimal one place further to the right and it thus becomes $0.012\dot{3}7\dot{2}$, which is coterminous with the first.

It is evident if two repeating decimals are coterminous they may be added or subtracted, and it is for this purpose that this reduction is made.

In particular cases multiplication or division of coterminous circulates may be easily effected, but in general

it is better to reduce the circulates to the form of common fractions before multiplying or dividing.

Some examples of addition and subtraction will now be given.

Required to add $4.8\dot{3}7\dot{9}$ and $12.257\dot{6}4\dot{3}$.

In the first number the repeating places begin with the order of hundredths and in the other with the order of ten-thousandths. Transferring the repeating set in the first number two places to the right we have $4.837\dot{9}3\dot{7}$, and writing the numbers in order for addition, we have

$$\begin{array}{r} 4.837\dot{9}3\dot{7}^{9} \\ 12.257\dot{6}4\dot{3}^{3} \\ \hline 17.095\dot{5}8\dot{1} \end{array}$$

In adding the first column at the right we find the sum to be 10, but it is apparent that were the next column written out the sum would be more than 10, and there would be, therefore, 1 to carry.

In any case, whether of addition or subtraction of circulates, after the decimals are made coterminous and written in proper order, one must determine by inspection of the numbers that have been omitted in the column or columns next at the right, whether there is one or more to be carried to the first column at the right. It will be found necessary in some cases to examine two or more of the omitted columns. Thus to add

$$\begin{array}{r} 0.2\dot{8}7\dot{6}^{8\,7} \\ 0.3\dot{1}4\dot{8}^{1\,4} \\ \hline 0.6\dot{0}2\dot{5}^{0} \end{array}$$

In this case the sum of the numbers belonging to the first column omitted is only 9, but looking at the numbers belonging to the second omitted column, their sum is more than 9, and would, therefore, cause 1 to be added to the first omitted column; and hence, cause 1 to be added to the first column at the right of the set expressed.

Again, suppose we have to add

$$\begin{array}{l} 0.\dot{5}\dot{3} \\ 0.\dot{4}\dot{6} \\ \hline 0.\dot{9}\dot{9} = 0.9 = 1 \end{array}$$

Or again,

$$\begin{array}{l} 0.\dot{2}78\dot{4} \\ 0.\dot{7}21\dot{5} \\ \hline 0.\dot{9}99\dot{9} = 0.\dot{9} = 1 \end{array}$$

Let it be required to subtract $0.\dot{4}\dot{6}$ from $0.\dot{5}\dot{3}$:

$$\begin{array}{l} 0.\dot{5}\dot{3} \\ 0.\dot{4}\dot{6} \\ \hline 0.\dot{0}\dot{7} \end{array}$$

EXERCISES.

(1). Reduce the fractions in the following list to the decimal form:

$$\tfrac{3}{4},\ \tfrac{19}{25},\ \tfrac{14}{35},\ \tfrac{7}{64},\ \tfrac{1}{125},\ \tfrac{2}{3},\ \tfrac{8}{9},\ \tfrac{16}{800}.$$

Ans. $.75,\ .76,\ .4,\ .109375,\ .08,\ .\dot{6},\ .\dot{8},\ .02.$

An error occurs in one of these answers, which the student should be able to detect.

(2). Reduce to a decimal form $\frac{7}{8}$, $\frac{9}{10}$, $\frac{19}{20}$, $\frac{36}{750}$, $\frac{19}{64}$, $\frac{1}{7}$, $\frac{4}{130}$, $\frac{4}{28376999}$, $\frac{4\frac{1}{2}}{25}$, $\frac{33\frac{1}{3}}{1280}$.

(3). Reduce to a common form .125, .875, .95, .048, .3125, .0288, $.\dot{5}$, $.5.\dot{5}$, $.0\dot{7}$, $.\dot{0}\dot{7}$, $.\dot{7}\dot{0}$, $.\dot{1}2\dot{1}$, $.0\dot{1}44\dot{2}$, and reduce to simplest terms.

(4). Reduce the following to coterminous circulates and add together:

$.01\dot{2}\dot{3}$, $.0\dot{4}\dot{5}$, $.0\dot{1}8\dot{3}$, $.0420\dot{1}2\dot{3}$.

REVIEW XVIII.

a. **To reduce a fractional number from the common to the decimal form,** divide the numerator by the denominator, expressing the quotient in decimal orders

b. **If the division** does not finally end without any remainder, the quotient may be expressed as a complex decimal, or as a repeating decimal, in which the same numbers are repeated in different orders of decimals.

c. **A repeating decimal** is also called a circulating decimal, or a circulate, and is indicated by placing a dot over the first and over the last of the series of figures which are repeated in

the expression, or if only one figure is repeated, over that one.

d. **To reduce a simple decimal to the common form,** express the denominator in the common form and reduce to the simplest terms.

e. **To reduce a repeating decimal to the common form of a fractional number,** multiply by 10, or 100, or 1000, and so on, according to the number of repeating figures in the expression, so that subtracting the given decimal from the product the difference will not contain any repeating portion. Then divide this remainder by a number 1 less than the multiplier, and the quotient expressed in the common fractional form will be the answer required.

MISCELLANEOUS EXERCISES.

In the first four of these exercises the given numbers are to be added. In the remaining portion the required operations are indicated.

(1).	(2).
985471022111795	873459911000684
888888888888888	999999999999999
989796959493929	878685848382818
665544332210999	776655443322110
555666444333221	666777555444332
666655557779991	555544446668880
777111222333449	777111222333449
234567901098765	123456789987654
876543209890123	987654321001234
778890153456789	778890123456789
444444444444444	333333333333333
221111112233333	110000001122222
777777777777777	888888888888888
681177236540728	792288347651839
742135984568814	641024873457703
666666666666666	777777777777777
732856670542475	843967781653586
241099887766554	129988776655443
333333333333333	444444444444444
280987942552074	169876831440963
333366667778888	222255556667777
555555555555555	666666666666666
786756453423120	897867564534231
345803957947840	234692846836729
469284683672448	469284683672448
125877351856713	236988462967824
656491247388250	545380246277139

(3).	487392641235	(4).	9871236894488134789
	337659278421		4338239754937172874
	478396017748		5577883344339977665
	237964512008		4321234567899876543
	573421642823		3255776198246397518
	777788889999		4444444444444444444
	428321291768		5555555556666665555
	4319234765		7777778888887771111
	28921776		9999999999999999999
	17762982		8888888888888888888
	5674329134		1234567899876543210
	39871646882		3333333333333333333
	867192123824		2123434545656767878
	328246124375		1299878876677655654
	34971062843		5948376218754837752
	75645832008		6667711114937825596
	6310897642		753123498978576432
	5581200034		35841247789668831
	462993375		2837435962738594
	649239735		152937485663784
	42810024		57798183542127
	8413439		3462167378425
	194876		213198765421
	23384		88778293555
	67849		2222222222
	934318		999999999
	4200184		48376425
	53793295		2141377
	430002185		35435129
	2467980136		9876543210044323730

(5). $876549321673 \times 293867514292$.

(6). $9983946672912 \times 928376514238$.

(7). $2192766493899 \times 832415673829$.

(8). $44883377556633 \times 33665577338844$.

(9). $9911228877334466 \times 6644337788221199$.

(10). $11335577992244 \times 224466883579$.

(11). $44229977553311 \times 975388664422$.

(12). $333444555666 \times 777888999222$.

(13). $777777656565 \times 666666494949$.

(14). $821973467927 \times 55556666$.

(15). $218794327933 \times 8888777755$.

(16). $999999999999 \times 555555555555$.

(17). $847638424896 \div 4096$.

(18). $48367284969696 \div 1728$.

(19). $8496625 \div 125$.

(20). $6125625 \div 125$.

(21). $(275 \times 84 - 32 \times 25) \times 20$.

(22). $275 \times 84 - 32 \times 35 \times 20$.

(23). $(24 \times 12 + 28 \times 16) \div 32$.

(24). $24 \times 12 + 28 \times 16 \div 32$.

(25). $(184 \times 36 - 144 \times 24) \div 72$.

(26). $184 \times 36 - 144 \times 24 \div 72$.

(27). What is the greatest common factor of 2432 and 4848? *Ans.*

(28). What is the greatest common factor of 2442 and 4218? *Ans.*

(29). What is the greatest common factor of 2652 and 4998? *Ans.*

(30). Find the greatest common factor of 1716, 2574, and 4368. *Ans.*

(31). Find the greatest common factor of 459, 1020, and 1275.

(32). Find the greatest common factor of 803, 949, 1314.

(33). Find the least common multiple of 5, 6, 7, 8, 9.

(34). Find the least common multiple of 18, 36, 63.

(35). Find the least common multiple of 16, 20, 32, 45.

(36). Find the least common multiple of 48, 72, 108, and 180.

(37). $(4\frac{1}{2} \times \frac{7}{8} \div \frac{1}{10} - 8\frac{2}{3} \div \frac{1}{6} \times 10\frac{1}{2}) \times \frac{1}{4}$. *Ans.* $8\frac{407}{672}$.

(38). $4\frac{1}{2} \times \frac{7}{8} \div \frac{1}{10} - 8\frac{2}{3} \div \frac{1}{6} \times 10\frac{1}{2} \times \frac{1}{4}$. " $19\frac{25}{168}$.

(39). $24\frac{1}{2} \times \frac{7}{8} \div \frac{1}{10} - (8\frac{2}{3} \div \frac{1}{6}) \times 10\frac{1}{2} \times \frac{1}{4}$. " $77\frac{7}{8}$.

(40). $(36 \div 18) \div 9$. " $\frac{2}{9}$.

(41). $36 \div 18 \div 9$. " 18.

(42). $(24\frac{1}{2} - (\frac{17}{5} \div 3\frac{1}{2}) \div \frac{8}{9} + \frac{3}{4} \times 12\frac{1}{2} \div \frac{3}{4} \div \frac{3}{2}) \times 8\frac{1}{2}$.

Ans. $273\frac{47}{140}$.

(43). $(19\frac{1}{4} \div 16\frac{1}{3} \times \frac{1}{7} - 76 \div 19 \times \frac{1}{5}) \times \frac{2}{3}$. " $2\frac{1}{6}$.

(44). $19\frac{1}{4} \div 16\frac{1}{3} \times \frac{1}{7} - 76 \div 19 \times \frac{1}{5} \times \frac{2}{3}$. " $\frac{3}{4}$.

(45). $\frac{3}{4} \times (20\frac{1}{2} \times \frac{36}{41} + 10\frac{2}{3} \times \frac{21}{8} - 52\frac{1}{2} \times \frac{6}{7})$.

(46). 8.08 — 8.0008.

(47). 125000 —.000125.

(48). 10 —.00001.

(49). $125 \times .00008$.

(50). $.0016 \times .0025$.

(51). $273.48 \times .0756$.

(52). 71.38025×2.4096.

(53). $.0044856 \div 1000$.

(54). $.000008 \div .002$.

(55). $.0008 \div .0000002$.

(56). $.0000002 \div .0008.$

(57). $12400 \div .0031.$

(58). $5.474558 \div .0325.$

(59). $217.568 \div .7854.$

(60). $1563.277 \div 426.$

(61). $102.4 \div 4096.$

(62). $4096 \div 102.4.$

(63). $100.25 \times \frac{36}{360} \times .08\frac{1}{2}.$

(64). $183.33 \times \frac{129}{360} \times \frac{9}{100}.$

(65). $25.50 \times \frac{589}{360} \times \frac{6}{100}.$

(66). $48.75 \times (1 + \frac{23}{360}) \times .07.$

(67). $10.075 \times (2 + \frac{73}{360}) \times \frac{4\frac{1}{2}}{100}.$

(68). $1000 \times \frac{125}{360} \times .08\frac{1}{2}.$

(69). $.25 \times (5 + \frac{75}{360}) \times .50.$

(70). $1500 \times \frac{93}{360} \times \frac{12}{100}.$

(71). $175.75 \times \frac{72}{360} \times .37\frac{1}{2}.$

(72). $33.33\frac{1}{3} \times \frac{50}{1\cdot6\frac{2}{3}} \times \frac{4\frac{1}{3}}{\cdot7} \times \frac{\cdot07}{9}.$

(73). $100.25 \times \frac{\cdot37\frac{1}{2}}{1\cdot25} \times \frac{0\cdot9\frac{1}{2}}{1\cdot90}.$

(74). $87.87\frac{1}{2} \times \frac{12\cdot8}{\cdot36} \div \frac{3\frac{1}{3}}{25}.$

(75). $875.692 \times \frac{4\cdot096}{15625} \times \frac{1\cdot8}{24}.$ *Ans.* .0172168052736.

(76). $(428\frac{1}{3} \times \frac{12\cdot5}{8\cdot75} - \frac{365}{4\cdot2} \times \frac{1\cdot47}{25}) \times \frac{5}{1\cdot25}.$

Ans. $2427.17\frac{19}{21}.$

(77). $428\frac{1}{3} \times \frac{12\cdot5}{8\cdot75} - \frac{365}{4\cdot2} \times \frac{1\cdot47}{25} \times \frac{5}{1\cdot25}.$ " $591.46\frac{10}{21}.$

(78). $184.7\frac{7}{9} \times \frac{1\cdot24}{31} \times \frac{365}{\cdot03\frac{1}{3}}.$ " $80932\frac{2}{3}.$

(79). $45.45 \times \frac{2\cdot5}{\cdot11\frac{1}{4}} \times \frac{8\cdot75}{12\cdot5}.$ " 707.

(80). $1728 \times \frac{18\cdot75}{26} \times \frac{3\frac{1}{4}}{7\cdot5}.$ " 2160.

PART II.

NUMBER USED IN THE MEASURE OF QUANTITY.
COMPOUND DENOMINATIONS.

PART II.

NUMBER USED IN THE MEASURE OF QUANTITY. COMPOUND DENOMINATIONS.

CHAPTER I.

Number and Quantity.

In the preceding part of this treatise only the simple operations upon numbers have been considered.

It will be found, in applying these operations to practical questions, that other considerations demand attention and study, and further on an explanation will be given of the principles useful in such cases. But in this part of the treatise it is intended to explain the various measures of quantities in common use, a knowledge of which is needed as a preliminary to the practical applications just alluded to.

The most frequent use of number is in the measure of quantity, and it is important to understand clearly the distinction between the terms number and quantity, which are often confounded.

Number, we have already learned, tells how many

things are considered, while *quantity tells how much is considered.*

Each of the terms yard, foot, acre, mile, pound, gallon, and many others, expresses a quantity which has been agreed upon, and is understood, as a standard of measure.

When it is required to measure the quantity of anything—that is, to tell how much there is—the standard of measure which has been agreed upon is applied to the thing to be measured, so that every portion shall be measured once and only once. The number of times this standard of measure can be so applied expresses the quantity to be measured. For instance, to find the length of a given line, suppose a yard is taken as a standard, and suppose this measure to be applied 6 times along the line, then 6 yards expresses the whole length of the line.

In a similar way the expressions "4 acres," "10 pounds," "3 gallons," and others similar in frequent use, are understood.

It may be remarked that in many cases the comparison is not made directly with the standard named, but by some intermediate means.

Thus 4 acres would not be compared directly with a single acre, but indirectly by means of a surveyor's chain.

It is apparent from the foregoing in what manner number is used with quantity, but the distinction between them should be clearly understood.

It is sometimes said that "quantity is anything that may be increased or diminished, or anything that may

be measured." It would be better to say that "*quantity may be affirmed* of whatever may be increased or diminished, or that may be measured." But even this would not be true, for, according to the usage of language, there may be an increase of number as well as an increase of quantity.

Thus we may say *many*, *more*, *most*, referring to number, or *much*, *more*, *most*, referring to quantity. In the first case more and most are as distinct from more and most in the second case, as "many" is distinct from "much."

Time, space, value (usually represented by money), and force (usually represented by weight or pressure), are the things the measures of which are most frequently used, and will be especially considered in the following chapters.

CHAPTER II.

Measures of Time.

TIME marks the succession of events, and years, months, and days are the most obvious natural divisions of time.

There are several periods, slightly differing, sometimes designated by the name year, but the tropical year is the one in ordinary use, and it includes the interval of time extending from one vernal equinox to another, and is very nearly the period required for the earth to revolve in its orbit about the sun.

The four seasons, Spring, Summer, Autumn, and Winter, also complete their rounds in the course of a year.

The month is derived from the interval of time occupied by the moon in its revolution about the earth.

The names of the months, and the number of days assigned to each, are indicated in the following list:

January	31 days.	June	30 days.
February—28, or in Leap year	29 days.	July	31 days.
March	31 days.	August	31 days.
April	30 days.	September	30 days.
May	31 days.	October	31 days.
		November	30 days.
		December	31 days.

Many have found help in remembering the number of days in the different months from the following lines:

> "Thirty days hath September,
> April, June, and November;
> All the rest have thirty-one
> Save February alone."

The week of seven days has been in use from a remote antiquity, and was probably derived from the month as a fourth part.

The day is marked by the successive rising and setting of the sun.

The division of the day into 24 hours seems also to be of very remote origin, having been in use by the Egyptians and Chaldeans. That is, they divided day and night each into 12 equal parts. It seems probable that this division into 12 parts was suggested by the division of the year into 12 parts, which had already led to the 12 signs of the zodiac.

The division of each hour into 60 minutes, and the division of each minute into 60 seconds, is perhaps of more modern origin.

The following table shows the various divisions of time named in the foregoing:

Time Table.

60 seconds	make	1 minute.
60 minutes	"	1 hour.
24 hours	"	1 day.
7 days	"	1 week.
30 days	"	1 calendar month.
365 days	"	1 common year.
366 days	"	1 leap year.
100 years	"	1 century.

An orderly arrangement of the divisions of time for the purpose of reckoning the dates of events is called a calendar.

The one in common use is called the Gregorian calendar, because first established by a decree of Pope Gregory.

This will now be explained.

But first it is important to understand clearly how the length of a day is reckoned.

Apparent noon is the instant at which, in our northern latitude, the shadow of any object caused by the sun, appears to fall due north.

A day is reckoned from noon to noon again, but as the apparent motion of the sun is not uniform, the length of the days would vary.

To avoid this result an imaginary sun, called the Mean Sun, is conceived, which shall appear to move on the celestial equator at a uniform rate, which shall make as many circuits as the real sun, and which, at some given moment, shall coincide in position with the real sun.

The time at which the Mean Sun would appear on the meridian is called mean noon, and the length of a day is reckoned from one mean noon to the next.

Such is the *length* of a day; but the day is usually reckoned to ***begin at midnight***, which is the moment half way between two consecutive noons.

Astronomers, however, reckon the day to begin at noon, and, of course, to end at the next noon following.

The average length of the year is 365 days, 5 hours, 48 minutes, and 49.7 seconds.

In the course of 4 years the annual excess of 5 hours, 48 minutes, and 49.7 seconds would amount to 23 hours, 15 minutes, 18.8 seconds, or only 44 minutes and 41.2 seconds less than 24 hours. By reckoning 1 extra day each 4th year—the 29th of February—the accumulated error is more than balanced, and a small error of deficiency is incurred. In 100 years this amounts to nearly three-fourths of a day. This again is nearly balanced by reckoning each 100th year, excepting each 400th year, as a common year. That is, the 100th years are not leap years, except the 400th years, which are leap years. By this reckoning the error in 400 years would be reduced to 2 hours 28 minutes.

In 10 times 400 years—that is, in 4000 years—this error would accumulate to 24 hours 40 minutes, and by reckoning the 4000th year as a common year, the error would be reduced to 40 minutes.

Recapitulation.

Each 4000th year is a common year.
Each of the other 400th years is a leap year.
Each of the other 100th years is a common year.
Each of the other 4th years is a leap year.

The Julian Calendar was established by Julius Cæsar 66 years B.C.

In this calendar every 4th year contained 366 days, the extra day being added in the month of February by reckoning the 6th day before the Kalends of March twice. Hence the year was called bissextile, from *bis* meaning twice, and *sextus*, sixth.

This calendar continued in use until the decree of

Pope Gregory establishing the new order, at the same time striking 10 days from the calendar. That is, the day following the 3d of October, 1582, was called the 14th.

This caused the vernal equinox to fall on the 21st of March, the same as it occurred in the year 325, the date of the Council of Nice.

The Gregorian Calendar was adopted at once in the Catholic countries of Europe, and gradually in all Christéndom, excepting Russia, where the Julian Calendar is still in use.

In the transition from the use of one calendar to that of the other, a date was frequently given by reference to both, respectively designated as Old and New Style.

In this case the figures expressing the days of the month according to one calendar, were written above those of the other. Thus, February $\frac{11}{22}$, 1732, indicated February 11, Old Style, or February 22, 1732, New Style.

CHAPTER III.

Measures of Space.

Space is that which contains all physical magnitudes. Space has three dimensions, length, breadth, and thickness. A line is space extending in length only. A surface is space extending in length and breadth only. A volume is space extending in length, breadth, and thickness. A solid is the body which occupies some volume.

SECTION I.

Measures of Length.

The standard measure of length in ordinary use in the United States is the yard. The meter, also authorized by the general government and by some of the States, will be subsequently mentioned. The length of the yard is determined by the length of a pendulum beating seconds.

Such a line is supposed to be divided into 391393 equal parts, and 360000 of these parts constitute a yard. The other divisions of length will be understood from the tables that follow.

Linear Measure.

12 inches (in.)	make	1 foot.........	ft.
3 feet	"	1 yard.........	yd.
5½ yards	"	1 rod	rd.
40 rods	"	1 furlong......	fur.
8 furlongs	"	1 statute mile..	mi.

Surveyors' Measure.

FOR MEASURING LAND.

7.92 inches make 1 link	l.
100 links, or 4 rods, make 1 chain..	ch.
80 chains make 1 mile...........	mi.

Circular Measure.

This is chiefly used in Navigation, Astronomy, and Geography.

The circumference of a circle is the curved line around it, every point of which is equally distant from the centre.

A degree is $\frac{1}{360}$ part of the circumference of a circle.

The subdivisions are shown in the following

Table.

60 seconds (″)	make	1 minute........	′
60 minutes	"	1 degree........	°
30 degrees	"	1 sign..........	S.
12 signs	"	1 circumference..	C.

As a circle may be larger or smaller, so a degree, or any other portion of one circumference, may be larger or smaller than the same portion of another circumference.

The longitude of a place is the distance of that place east or west from some established meridian, and may be reckoned either in degrees, minutes, and seconds, or in hours, minutes, and seconds of time. The difference of longitude of two places is the difference of those distances. In this country longitude is reckoned

from the meridian of Washington, and sometimes from that of Greenwich, England.

As the sun appears to move around the earth, or 360° in 24 hours, it will appear to move over 15° in each hour. It will require 4 minutes to move over 1°. In 1 minute it will move over $\frac{1}{4}$ of a degree, which is 15′. In 1 second of time it moves over 15″ of arc.

As the sun appears to move from east to west, it will appear on any eastern meridian sooner than on a western—according to the rate just described.

As the hours of the day are reckoned from noon, it follows that clock time is not the same in two places, one east and the other west of a given point; and the difference in the clock time of two places amounts to 1 hour for each 15° of difference of longitude; or 1 minute of time for 15 minutes of longitude, and 1 second of time for 15 seconds of longitude.

Latitude is distance north or south of the earth's equator, reckoned in degrees, minutes, and seconds, of the earth's circumference. A degree of latitude varies from 68.72 miles near the equator, to 69.34 miles near the poles. The mean length of a degree of latitude is reckoned at 69.16 miles.

A degree of longitude varies from 69.16 miles at the equator, to 0 at the poles.

Miscellaneous Measures of Length.

The following are sometimes used:

3 barleycorns make 1 inch. (Used by shoemakers.)

4 inches make 1 handbreadth. (Used in measuring the height of horses.)

9 inches make 1 span.
6 feet make 1 fathom.
1.15 statute miles make 1 geographical mile.
3 geographical miles make 1 league.

SECTION II.

Square Measures, or Measures of Surface.

The surface contained within the lines which bound it, is called the area.

The standard measure for surfaces is a square or a surface equal in length and breadth.

A "foot square" means a surface a foot in length and the same in breadth. A "square foot" means a surface of the same amount, but it may have any shape whatever. It may have only three sides, or it may have four or any greater number of sides.

"Two feet square" means a surface two feet in length and two feet in breadth; and it is easy to see that from such a surface four equal squares, each containing a square foot, may be formed. The following diagram, Fig. 1, will illustrate the foregoing.

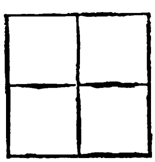


Fig. 1.
"2 feet square," containing 4 square feet.

In like manner a surface "3 feet square" contains 9 square feet.

Figure 2 illustrates a surface "3 feet square," and Figure 3 illustrates "3 square feet."

So 1 foot square contains 144 square inches, because it can be divided each way into 12 equal parts, forming 12 rows of 12 small squares, each 1 inch in length and breadth.

A flat surface of uniform length and uniform breadth is called a rectangle, and it is easy to see that the area of a rectangle is expressed by the product of the number of units in length multiplied into the number of units in breadth.

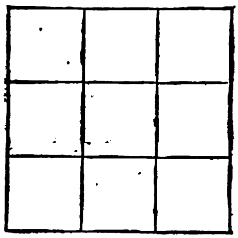


FIG. 2.
Three feet square.

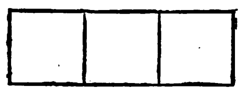


FIG. 3.
Three square feet.

Thus, if a rectangle is 2 feet wide and 3 feet long, it may be formed into 2 sections one way, and 3 sections the other way, so as to form 2 rows of squares, and 3 squares in each row, each square being one foot in length and breadth. Obviously the whole area would be 2×3 square feet. And whatever the length and breadth, any rectangle could be formed into squares of some dimension in a similar manner.

Table of Square Measures.

144	square	inches (sq. in.)	make	1	square foot...	sq. ft.
9	"	feet	"	1	" yard..	sq. yd.
$30\frac{1}{4}$	"	yards	"	1	" rod...	sq. rd.
40	"	rods	"	1	rood........	R.
4	roods		"	1	acre........	A.
640	acres		"	1	square mile..	sq. mi.

Surveyors' Square Measure.

This is used especially in computing the area of land.

Table.

625 square links (sq. ls.)	make	1 pole...........	P.
16 poles	"	1 square chain...	sq. ch.
10 square chains	"	1 acre...........	A.
640 acres	"	1 square mile....	sq. mi.
36 square miles	"	1 Township......	Tp.

A mile square is often called a section of land.

SECTION III.

Measures of Volume. Cubic Measure.

This is used in measuring solid bodies, as wood, stone, earth, etc.

The standard of measure is a cube or body of equal length, breadth, and thickness.

An inch cube is 1 inch in length, 1 inch in breadth, and 1 inch in depth. A foot cube is 1 foot in length, breadth, and depth.

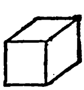


FIG. 1.
1 *inch cube.*

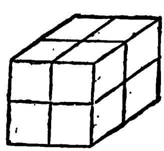


FIG. 2.
2 *inches cube.*

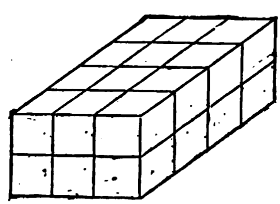


FIG. 3.
Block 2 *inches in height,* 3 *inches in width,* 4 *inches in length.*

The cubic or solid contents of a body are found by multiplying together the units of length, breadth, and depth.

It is easy to see, from Fig. 2, that a block of 2 inches cube would form, by the sections indicated, 8 blocks of cubic inches, or $2 \times 2 \times 2$ cubic inches.

Also the sections indicated in Fig. 3 would form $2 \times 3 \times 4 = 24$ blocks of cubic inches.

In each case, the number of cubic units is the product of the numbers of units in length, breadth, and depth, multiplied together.

Table.

1,728 cubic inches (cu. in.) make 1 cubic ft....	cu. ft.
27 cubic feet make 1 cubic yard..........	cu. yd.
40 cubic feet of round timber, or 50 cubic feet of hewn timber make 1 ton or load.	T.
16 cubic feet make 1 cord foot...........	cd. ft.
8 cord feet, or 128 cubic feet make 1 cord of wood...........................	cd.
$24\frac{3}{4}$ cubic feet make 1 perch of stone or masonry...........................	pch.

"40 cubic feet of round timber," means timber that will produce 40 cubic feet when hewed or sawed into square timber. It is supposed to contain 50 cubic feet, but that it will lose 10 cubic feet ($\frac{1}{5}$) in the hewing or sawing. Its value is then only that of 40 cubic feet of hewn timber, but its weight or volume is equal to that of 50 cubic feet of hewn timber.

Dry Measure.

This is used in measuring articles of a dry nature, as grain, fruit, seeds, salt, etc.

The bushel is the standard of measure, and the other denominations are given in the following

Table.

2 pints (pt.)	make	1 quart	qt.
8 quarts	"	1 peck	pk.
4 pecks	"	1 bushel	bu.

A cylindrical box, 8 inches in depth, and $18\frac{1}{2}$ inches in diameter, holds a bushel. Such a box contains 2150.42 cubic inches. Four quarts (which are an eighth part of the bushel) contain 268.8 cubic inches.

Wine Measure.

Wine Measure, or Liquid Measure, is used in measuring liquids, as oils, liquors, milk, molasses, etc.

The gallon is the standard of liquid measure, and the other denominations are shown in the table following.

Table.

4 gills (gi.)	make	1 pint	pt.
2 pints	"	1 quart	qt.
4 quarts	"	1 gallon	gal.
$31\frac{1}{2}$ gallons	"	1 barrel	bbl.
2 barrels or 63 gals.	make	1 hogshead	hhd.

A gallon contains 231 cubic inches. The difference

in the volume of dry and liquid measures of the same name, are shown as follows:

	Cubic inches in		
	4 qts.	1 qt.	1 pt.
Dry measure.........	$268\frac{4}{5}$	$67\frac{1}{5}$	$33\frac{3}{5}$
Wine measure........	231	$57\frac{3}{4}$	$28\frac{7}{8}$

Apothecaries' Fluid Measure.

This is used in measuring all liquids for the compounding of medical prescriptions.

60 minims (♏)	make	1 fluid drachm...	f ʒ
8 f ʒ	"	1 fluid ounce.....	f ℥
16 f ℥	"	1 pint...........	O.
8 O.	"	1 gallon.........	cong.

CHAPTER IV.

Weights, or Measures of Force.

WEIGHT is the measure of the force of gravity, but it serves to show the quantity of matter contained in a body.

The United States standard of weight is derived from the weight of a cubic inch of distilled water at a temperature of 62° Fahrenheit, in a vacuum. This weight is reckoned as 252.458 grains.

Troy Weight.

This is used in weighing gold, silver, and jewels, and the materials used in philosophical experiments.

The pound is the standard, and contains 5760 grains, of which 252.458 measures the weight of a cubic inch of distilled water, as already described.

Other divisions of the pound are shown in the following

Table.

24 grains (gr.)	make	1 pennyweight..	pwt.
20 pennyweights	"	1 ounce........	oz.
12 ounces	"	1 pound........	lb.

Apothecaries' Weight.

This is used in mixing medical prescriptions. In other cases drugs are bought and sold by commercial weight.

20 grains (gr.)	make	1 scruple........	℈
3 ℈	"	1 drachm.......	ʒ
8 ʒ	"	1 ounce.........	℥
12 ℥	"	1 pound........	℔

The pound, ounce, and grain are the same as those of Troy weight.

Avoirdupois, or Commercial Weight.

This is used for the ordinary purposes of commercial transactions in weighing, not including those mentioned for the other tables of weights.

16 drachms (dr.)	make	1 ounce..........	oz.
16 oz.	"	1 pound..........	lb.
100 lbs.	"	1 hundred weight.	cwt.
20 cwt., or 2000 lbs.,	make	1 ton........	T.

A pound avoirdupois weight contains 7000 grains, of the kind used in Troy weight, and is, therefore, heavier than a pound Troy. The ounce Troy contains one-twelfth part of 5760—that is, 480 grains—while the ounce avoirdupois contains one-sixteenth of 7000—that is, 437½ grains. Hence the ounce Troy is heavier than the ounce avoirdupois. Accordingly a pound of feathers weighs more than a pound of gold, but an ounce of gold weighs more than an ounce of feathers.

It should be remarked that the true weight of a body is not usually indicated when weighed in the air, but may be computed when the density of the body and the density of the air are both known. The error is so small, however, in all ordinary cases, that it may safely be neglected.

CHAPTER V.

Miscellaneous Tables.

12 things	make	1 dozen.
20 things	"	1 score.
12 dozen	"	1 gross.
12 gross	"	1 great gross.

24 sheets of paper	make		1 quire.
20 quires	"	"	1 ream.
2 reams	"	"	1 bundle.
5 bundles	"	"	1 bale.

Sheets of paper are of several sizes, and these again differ in the various branches of business. Paper to be used for the printing of books is formed of the sizes named in the following table, in which the number of inches in the length and breadth of a sheet are stated, and the letter x placed between the figures of the two numbers.

Thus 19 x 24 means 19 inches in breadth and 24 inches in length.

Table.

Medium sheet measures............	19 x 24.
Super-Royal sheet measures........	22 x 28.
Medium-and-half sheet measures....	24 x 30.
Double Medium sheet measures....	24 x 38.
Double Royal sheet measures......	26 x 40.
Double Super-Royal sheet measures	28 x 42.
Double Imperial sheet measures....	32 x 46.

These dimensions, however, being established by custom, are liable to change.

The sizes of books are named according to the number of leaves into which the sheets are folded to form the pages.

A sheet	folded in	2	leaves	forms the	folio size.
A sheet	"	4	"	"	quarto or 4to.
A sheet	"	8	"	"	octavo, or 8vo.
A sheet	"	12	"	"	duodecimo, or 12mo.
A sheet	"	16	"	"	16mo.
A sheet	"	18	"	"	18mo.
A sheet	"	24	"	"	24mo.
A sheet	"	32	"	"	32mo.

Thus a book is said to be a medium 8vo when one leaf is the eighth part of a medium sheet.

CHAPTER VI.

The Metric System.

In the foregoing tables of various kinds, the subdivisions and multiples of the standard units are not formed according to any uniform system, but in most cases follow the usage which originated in a more primitive condition of society. Thus the weight of a grain (of which 24 make 1 pennyweight) was derived from a grain of wheat. And it is said that the length of the British yard was originally estimated by the length of the arm of King Henry I.

In other countries, still more diverse systems of measures are in use, and the commerce between different peoples, as well as the traffic carried on among the same people, is often hampered by the inconvenience of the complex measures in use.

Recently, efforts have been made to introduce a more simple system, and Congress has authorized in the United States the metric system, previously established in France and subsequently adopted by Spain, Belgium, and Portugal, and authorized in Great Britain, Holland, Norway, Sweden, and other countries.

The metric system is based upon the length of the meter, which was intended, when first established, to

be the ten-millionth part of the quadrant of the earth's circumference, estimated from the equator to the pole.

The length of the meter was established in France in 1799, and a platinum bar indicating the length was deposited in the national archives. It is now known that the meter thus determined is slightly less than the ten-millionth part of a quadrant of the earth's circumference, but it is still retained in use, and answers every required purpose.

The length of the meter, expressed in inches, is 39.37.

Multiples of the meter, and of other units of measure, are named by using prefixes derived from the Greek numerals—deka (ten), hecto (one hundred), kilo (one thousand), myria (ten thousand). Thus:

Dek'ameter (dkm.) = 10 meters.
Hec'tometer (hm.) = 100 meters.
Kil'ometer (km.) = 1000 meters.
Myr'iameter (myrm.) = 10,000 meters.

Subdivisions or submultiples of the meter, and of other units of measure, are named by using prefixes derived from the Latin numerals—decem (ten), centum (one hundred), mille (one thousand). Thus:

Dec'imeter (dcm.) = $\frac{1}{10}$ meter.
Cen'timeter (cm.) = $\frac{1}{100}$ meter.
Mil'limeter (mm.) = $\frac{1}{1000}$ meter.

The unit of surface is a square dekameter, called an are.

Centare (ca.) = $\frac{1}{100}$ are.
Are (ar.) = 1 square dekameter = 119.6 sq. yd.
Hectare (ha.) = 100 ares.

The unit of capacity is the cubic decimeter, called a *liter* (*leeter*).

Milliliter (ml.) = $\frac{1}{1000}$ liter.
Centiliter (cl.) = $\frac{1}{100}$ liter.
Deciliter (dcl.) = $\frac{1}{10}$ liter = 0.908 dry qt. or 1.0567 liquid qt.
Liter (l.) = 1 cubic dekameter.
Dekaliter (dk.) = 10 liters.
Hectoliter (hl.) = 100 liters.
Kiloliter (kl.) = 1000 liters.

Ordinarily the cubic centimeter (c.c.) is used instead of the milliliter; and the cubic meter, instead of the kiloliter, is used as a unit of measure for solid bodies, and is called a stere (pronounced *stare*). Thus we should say a kiloliter or cubic meter of wine, or of water, while we should say a stere of sand or a stere of hay.

The unit of weight is the weight of a cubic centimeter of pure water weighed in a vacuum at the temperature of 4° Centigrade or 39.2° Fahrenheit, the temperature of greatest density. This weight is called a *gramme*.

Milligramme (mg.) = $\frac{1}{1000}$ gramme.
Centigramme (cg.) = $\frac{1}{100}$ gramme.
Decigramme (dcg.) = $\frac{1}{10}$ gramme.

Gramme (g.) = weight of cubic centimeter of water = 15.4322 grains.

Dekagramme (dkg.) = 10 grammes.
Hectogramme (hg.) = 100 grammes.
Kilogramme (kg.) = 1000 grammes.
Myriagramme (myrg.) = 10,000 grammes.
Quintal (Q.) = 100,000 grammes.
Tonneau (T.) = 10,000,000 grammes.

In practice, the multiples of tens, and the subdivisions of tenths, are but little used, as in the case of the reckoning of money (United States) but little use is made of dimes and eagles, but chiefly of dollars and cents, and, in accounts, of mills.

So the following tables show the multiples and submultiples mostly used.

Weight.

10 milligrammes = 1 centigramme.
100 centigrammes = 1 gramme.
100 grammes = 1 hectogramme.
10 hectogrammes = 1 kilogramme.

Length.

10 millimeters = 1 centimeter.
100 centimeters = 1 meter.
1000 meters = 1 kilometer.

Surface.

100 square meters = 1 are.
100 ares = 1 hectare.

Capacity.

1000 cubic centimeters = 1 liter.
100 liters = 1 hectoliter.

The equivalents of the metric measures, expressed in common measures, have been stated, but it is often convenient to know the values of the common measures expressed in the metric system.

These values may be easily deduced from the foregoing tables.

Thus 1 centimeter = 0.3937 inches, and obviously 1 inch = $\frac{1}{0.3937}$ centimeters = 2.54 centimeters.

In a similar way, the other values named in the following table may be obtained.

An inch = 2.54 centimeters.
A foot = 0.3048 meter.
A yard = 0.9144 meter.
A rod = 5.029 meters.
A mile = 1.6093 kilometers.
A sq. inch = 6.452 sq. centimeters.
A sq. foot = 0.0929 sq. meter.
A sq. yard = 0.8361 sq. meter.
A sq. rod = 25.29 sq. meters.
An acre = 0.4047 hectare.
A sq. mile = 259 hectares.
A cu. inch = 16.39 cu. centimeters.
A cu. foot = 0.02832 cu. meter.
A cu. yard = 0.7646 cu. meter.
A cord = 3.624 steres.
A liquid quart = 0.9465 liter.
A gallon = 3.786 liters.

A dry quart = 1.101 liters.
A peck = 8.811 liters.
A bushel = 35.24 liters.
An ounce av. = 28.35 grammes.
A pound av. = 0.4536 kilogramme.
A ton = 0.0072 tonneau.
A grain = 0.0648 gramme.
An ounce, Troy = 31.104 grammes.
A pound, Troy = 0.3732 kilogramme.

The learner should verify the foregoing results by making the computations. In one instance an error may be found. Can any other error be found?

CHAPTER VII.

Money, or Measures of Value.

MONEY, strictly speaking, consists of coin or metal stamped by authority of the government, and is used to represent the values of commodities exchanged in the traffic of people, one with another.

Pieces of paper, on which are printed promises to pay, also stamped by authority of government, are often included under the name money, and are often used instead.

In the United States the dollar is established as the unit of value. Other denominations of United States money are mentioned in the following

Table.

10 mills (m.)	make	1 cent......	ct.
10 cents	"	1 dime......	d.
10 dimes	"	1 dollar.....	$
10 dollars	"	1 eagle.....	E.

The mill is only used in computation; there is no coin of the value of a single mill.

Gold is coined in double eagles ($20), eagles ($10), half eagles ($5), and quarter eagles ($2½), three dollar, and dollar pieces.

Silver is coined in dollar, half, and quarter dollars, 20-cent pieces, and in dimes.

Nickel and copper are coined in half dimes (or 5 cents) and cents.

Gold coin contains 90 parts of gold and 10 parts of silver.

Silver coin contains 90 parts of silver and 10 parts of copper.

Nickel coin contains 25 parts nickel and 75 parts copper.

Usually, United States money is reckoned only in dollars and cents. Thus 25 eagles, 7 dollars, 4 dimes, and 3 cents would ordinarily be reckoned as 257 dollars and 43 cents, and written \$257.43. As the dollar is the unit of value, the cents are reckoned as so many hundredths of a dollar, and the decimal point placed between the figures of the dollars and cents.

It will also be noticed that the sign \$, which indicates dollars, is placed at the left of the figures instead of at the right. This sign is supposed to be a contraction of the letters U. S. (for United States), the U being placed upon the S.

The number of cents in any case may be expressed as the fractional part of a dollar in hundredths, either in the common or in decimal form. Thus 4 dollars and 4 cents may be expressed \$4.04, or $\$4\frac{4}{100}$; and 25 dollars and 15 cents may be expressed \$25.15, or $\$25\frac{15}{100}$.

In Canada the denominations of money are in part similar to those of the United States. The silver coins are the shilling, or 20-cent piece, the dime, and half dime.

The value of the 20-cent piece is $18\frac{2}{3}$ cents United States money, the dime $9\frac{1}{3}$ cents, the half dime $4\frac{2}{3}$ cents.

The denominations of money used in Great Britain are shown in the table below. The pound sterling is the unit.

English Money.

4 farthings (qr.)	make	1 penny.....	d.
12 pence	"	1 shilling....	s.
20 shillings	"	1 pound.....	£

The value of a pound sterling, in United States money, is $4.84, the shilling is $0.242, the penny is $\$0.02\frac{1}{60}$, and the farthing is $\$0.00\frac{121}{240}$.

In France the unit of money value is the franc, and the other denominations may be seen in the following table:

French Money.

10 millimes	make	1 centime.
100 centimes	"	1 franc.

The gold coin is the Louis, or 20-franc piece; the silver coins are the franc and demi-franc (half-franc).

1 franc = $0.192.
1 centime = $0.00192.
1 millime = $0.000192.

The denominations and values of moneys used in many other countries will be found in the table in the Appendix at the close of the volume.

CHAPTER VIII.

Compound Denominations and their Reductions.

IT is often convenient to use two or more kinds of measures to express any given quantity.

A given tract of land may contain 10 acres, 2 roods, 5 square rods, and such a statement conveys a clearer idea of the actual quantity than the statement that the field contains 1685 square rods, though the two statements are equivalent. The explanation is, that the notion of a small number of large magnitudes is more easily apprehended than that of a great number of small magnitudes.

When the measure of a quantity is expressed in two or more denominations, these are called *compound.*

Sometimes the numbers used to enumerate compound denominations are called compound, but, strictly speaking, number is always simple, and there can be no such things as "compound numbers."

It is often desirable to change the measure of a quantity stated in one denomination to an equivalent in some other. This is called a *reduction of denominations.*

One denomination is said to be higher than another

when the unit of that denomination is more than one of the other. Thus the foot is of a higher denomination than the inch, and the mile is of a higher denomination than the yard.

Changing from a higher denomination to a lower is called *reduction descending;* changing from a lower to a higher denomination is called *reduction ascending.*

To change a number of one denomination to one of a lower denomination, multiply by the number which denotes the value of a given unit expressed in the lower denomination. Thus, to reduce 6 feet to inches; multiply by 12, and the product 72 expresses the number of inches equivalent to 6 feet.

To change a number of a given denomination to one of a higher, divide by the number of units of the lower denomination contained in one of the higher.

Thus, to reduce 72 inches to feet, divide by 12 (because 12 inches make 1 foot), and the quotient 6 expresses the number of feet equivalent to 72 inches.

The above precepts are so evident that they need no special explanation.

If the lower denomination is separated by one or more orders from the higher, we may reduce the given denomination successively to each intervening denomination. Thus, to reduce 10 miles to feet, we may reduce them first to furlongs (multiplying by 8), then to rods (multiplying by 40), then to yards (multiplying by $5\frac{1}{2}$), then to feet (multiplying by 3), as follows:

$$\begin{array}{r l}
10 & \text{mi.} \\
8 & \\
\hline
80 & \text{fur} \\
40 & \\
\hline
2)3200 & \text{rd.} \\
5\tfrac{1}{2} & \\
\hline
16000 & \\
1600 & \\
\hline
17600 & \text{yd.} \\
3 & \\
\hline
52800 & \text{ft.}
\end{array}$$

Or we may multiply at once by the number of feet contained in a mile, that is, by 5280.

$$10 \times 5280 = 52800, \text{ as before.}$$

It may be noticed that the first product above expresses furlongs, the second rods, the third yards, and so on, and it is worth while to inquire how the process of multiplication is carried on.

It is not correct to say that 10 miles are multiplied by 8 furlongs, because the multiplier only denotes a number of times (operations), not a number of objects or things. Properly speaking, 8 furlongs are multiplied by 10 (not 10 miles), and the reasoning is, "If there are 8 furlongs in 1 mile, then in 10 miles there are 10 times 8 furlongs; that is, 80 furlongs."

And in the second multiplication the multiplicand is really 40 rods; but the figures are written under

those of the multiplier because it happens to be convenient.

Again, to reduce $\frac{1}{4}$ mile to feet:

$$\begin{array}{lll} \frac{1}{4} \times 8 & = 2 \\ 2 \times 40 & = 80 \\ 5\frac{1}{2} \times 80 & = 440 \\ 3 \times 440 & = 1320 \text{ ft.} = \textit{Ans.} \end{array}$$

Or again, to reduce .0024 miles to feet:

$$\begin{array}{rl} .0024 & \\ 8 & \\ \hline .0192 & \text{fur.} \\ 40 & \\ \hline 2).7680 & \text{rd.} \\ 5\frac{1}{2} & \\ \hline 3840 & \\ 384 & \\ \hline 4.224 & \text{yd.} \\ 3 & \\ \hline 12.672 & \text{ft.} = \textit{Ans.} \end{array}$$

To reduce 1440 hours to weeks:

$$\begin{array}{l} 24)1440(60 \text{ da.} \\ \quad\;\; \underline{144} \\ \qquad\;\; 0 \end{array} \qquad\qquad \begin{array}{l} \underline{7)60} \\ \;\; 8\frac{4}{7} \text{ wk. } \textit{Ans.} \end{array}$$

Substantially the same principles apply when the measure of a quantity is expressed in several denominations.

Thus, to reduce 3 lb. 4 oz. 10 pwt. 17 gr. to grains. The process is as follows:

```
lb.  oz.  pwt.  gr.
  3   4   10    17
 12
 ---
 36 oz.
  4 oz.
 ---
 40 oz.
 20
 ----
 800 pwt.
  10 pwt.
 ----
 810 pwt.
  24
 -----
 3240
1620
-----
19440 gr.
   17 gr
-----
19457 gr. = Ans.
```

In this case it is seen that 3 lb. = 36 oz., to which 4 oz. are added, because both form parts of one quantity. For a similar reason 10 pwt. are added to 800 pwt. and 17 gr. to 19440 gr.

If it is required to reduce a given denomination to higher denominations, then any remainder that occurs at the end of a division may be left to represent that particular denomination. Thus, to reduce 1445 feet to

higher denominations. First divide by 3 to reduce to yards, then by $5\frac{1}{2}$, and so on.

$$
\begin{array}{l}
3\,)\underline{1445}\text{ ft.} \\
5\frac{1}{2}\,)\,481\text{ yd.} + 2\text{ ft. remaining.} \\
\underline{2}\quad\;\; \underline{\;\;2} \\
11\,)\underline{962} \\
\quad 40\,)\underline{87}\text{ rd.} + 5\text{ half yds.} = 2\frac{1}{2}\text{ yd. remaining.} \\
\qquad\quad 2\text{ fur. } 7\text{ rd. } 2\frac{1}{2}\text{ yd. } 2\text{ ft.}
\end{array}
$$

or 2 fur. 7 rd. 3 yd. 0 ft. 6 in., since $\frac{1}{2}$ yd. is equal to 1 ft. 6 in.

It may be remarked that in dividing by $5\frac{1}{2}$ it is convenient to multiply dividend and divisor by 2, and thus the dividend 481 yd. becomes 962 half yards, and the remainder 5 denotes half yards, and equals $2\frac{1}{2}$ yd., or 2 yd. 1 ft. 6 in., to which adding 2 ft., the result is 3 yd. 0 ft. 6 in.

If it is required to reduce a given denomination to any single higher denomination, then any remainder at the end of a division is expressed as a fractional quotient, either common or decimal.

For example, to reduce 1445 feet to miles, or to some part of a mile, divide first by 3, as in the previous case, then by $5\frac{1}{2}$, and so on.

$$
\begin{array}{l}
3\,)\underline{1445} \\
5\frac{1}{2}\,)\,481\frac{2}{3}\text{ yd.} \\
\underline{2}\quad\;\; \underline{\;\;\;\;2} \\
11\,)\underline{963\frac{1}{3}} \\
\qquad\; 87\frac{6\frac{1}{3}}{11} = 87\frac{19}{33}\text{ rd.}
\end{array}
$$

$$40)\underline{87\tfrac{19}{33}}\text{ rd.}$$

$$2\frac{7\frac{19}{33}}{40} = 2\tfrac{250}{1320} = 2\tfrac{25}{132}\text{ fur.}$$

$$8)\underline{2\tfrac{25}{132}}\text{ fur.}$$

$$\frac{2\frac{25}{132}}{8} = \tfrac{289}{1056}\text{ mi.}$$

But it would be easier to divide at once by the number of feet in a mile; that is, by 5280.

$\frac{1445}{5280} = \frac{289}{1056}$, the same as before.

If there are several denominations required to be reduced to the fractional part of a higher denomination, it will be best to reduce the given denominations to the lowest one of these, and then divide by the number of units of this denomination contained in a unit of the higher denomination required.

For example, to reduce 5 fur. 21 rd. 4 yd. 2 ft. to the fractional part of a mile:

$$\begin{array}{r l}
5\text{ fur.} & 21\text{ rd.}\quad 4\text{ yd.}\quad 2\text{ ft.} \\
\underline{40} & \\
2)\overline{221} & \\
\underline{5\tfrac{1}{2}} & \\
110\tfrac{1}{2} & \\
1105 & \\
\underline{4} & \\
1219\tfrac{1}{2} & \\
\underline{3} & \\
3660\tfrac{1}{2} & \quad \frac{3660\frac{1}{2}}{5280} = \frac{7321}{10560}.
\end{array}$$

The total number of feet in the given quantity is

thus found to be $3660\frac{1}{2}$, and dividing this by the number of feet in a mile, the result is

$$\frac{3660\frac{1}{2}}{5280} = \frac{7321}{10560}.$$

This can be changed to the decimal form, or the operation could be performed as follows:

3	2.0000
$5\frac{1}{2}$	4.6666+
40	21.848484+
8	5.5462121+
	.6932745+

Again, to reduce 3 pk. 5 qt. 1 pt. to the decimal fractional part of a bushel:

2	1.0
8	5.5
4	3.6875
	.921875. =

EXERCISES IN REDUCTION.

(1). Reduce 8 mi. 5 fur. 25 rd. 4 yd. 2 ft. 10 in. to inches. *Ans.* 551608 in.

(2). Reduce 6 S. 22° 48′ 48″ to seconds. *Ans.* 730128.

(3). Reduce 15 A. 1 R. 29 sq. rd. 4 sq. ft. to square inches. *Ans.* 96795252 sq. inches.

(4). Reduce 148 A. 5 sq. ch. 10 P. to square links.
Ans. 14856250 square links.

(5). Reduce 4 cd. 4 cd. ft. 4 cu. ft. 4 cu. in. to cubic inches. *Ans.* 1002244 cubic inches.

(6). Reduce 15 bu. 1 pk. 6 qt. to pints.
Ans. 988 pints.

(7). Reduce 40 gal. 3 qt. to gills. *Ans.* 1304 gills.

(8). Reduce 10 cong. 7 O. 12 f. ℥, 6 f. ʒ to minims.
Ans. 674280 minims.

(9). Reduce 48 lb. 10 oz. 15 pwt. 20 gr. to grains.
Ans.

(10). Reduce 24.10 ℥, 6 ʒ, 2 ℈, 10 gr. to grains. *Ans.*

(11). Reduce 11 T. 78 lb. 10 dr. to drachms. *Ans.*

(12). Reduce 10 lb. Troy weight to grains. *Ans.*

(13). Reduce 57600 gr. to pounds and ounces avoirdupois.

(Since there are $437\frac{1}{2}$ grains in an ounce avoirdupois, then in 57600 grains there will be $\frac{57600}{437\frac{1}{2}}$ ounces avoirdupois; but $\frac{57600}{437\frac{1}{2}} = 131\frac{23}{35}$. Dividing again by 16, because 16 ounces in a pound, we have $\frac{131\frac{23}{35}}{16}$ oz. $= 8$ lb. $3\frac{23}{35}$ oz.)

(14). Reduce 35 lb. Troy to pounds and ounces avoirdupois. *Ans.*

(15). Reduce 20 lbs. avoirdupois to pounds and ounces Troy weight. *Ans.*

(16). Reduce $\frac{7}{8}$ lb. avoirdupois to ounces. *Ans.*

(17). Reduce $\frac{7}{8}$ lb. Troy to ounces. *Ans.*

(18). Reduce $\frac{7}{8}$ yd. to inches. *Ans.*

(19). Reduce .12 ton to pounds. *Ans.*

(20). Reduce .9 bushel to pints. *Ans.*

(21). Reduce $17\frac{1}{2}$ lbs. Troy to pennyweights. *Ans.*

(22). Reduce .125 pound Troy to ounces and pennyweights. *Ans.*

(23). Reduce .125 lb. avoirdupois to ounces. *Ans.*

(24). Reduce .175 ton to pounds. *Ans.*

(25). Reduce 1 township to square inches. *Ans.*

(26). Reduce 1 sq. mi. to acres, also to square rods. *Ans.*

(27). Reduce .01 section to square chains. *Ans.*

(28). Reduce 1 circumference to seconds. *Ans.*

(29). Reduce 5 yd. 2 ft. $10\frac{1}{2}$ in. to the decimal of a rod. *Ans.* $1.08\frac{2}{3}$ rods.

(30). Reduce 2885 sq. ft. to the decimal of an acre. *Ans.*

(31). Reduce 187 pints to gallons. *Ans.*

(32). Reduce 2100 inches to yards. *Ans.*

(33). In 1000 cu. ft. how many cords? *Ans.*

(34). In $1457\frac{1}{2}$ cu. ft. how many cords? *Ans.*

(35). Reduce 5 lb. 10 oz. 10 pwt. to pounds and the decimal of a pound. *Ans.*

(36). Reduce 39 rd. 5 yd. 1 ft. 6 in. to the fractional part of a mile, also to furlongs. *Ans.*

(37). Reduce 1285 in. to the fractional part of a mile. *Ans.*

(38). Reduce 20 kilometers to miles and feet. *Ans.*

(39). Reduce 85.12 meters to feet and inches. *Ans.*

(40). Reduce 40 hectares to acres. *Ans.*

(41). Reduce 163.9 cu. centimeters to cu. inches. *Ans.*

(42). Reduce 94.65 liters to gallons. *Ans.*

(43). Reduce 1225 liters to dry quarts. *Ans.*

(44). Reduce 45.36 kilogrammes to pounds avoirdupois. *Ans.*

(45). Reduce 52.86 liters to bushels. *Ans.*
(46). Reduce 8000 kilogrammes to pounds Troy. *Ans.*
(47). Reduce 1800 square meters to square rods. *Ans.*
(48). Reduce 10 cd. 60 cu. ft. to steres. *Ans.*
(49). Reduce 8 rd. 3 yd. 2 ft. to meters. *Ans.*
(50). Reduce 10 gal. 3 qt. 1 pt. to liters. *Ans.*

CHAPTER IX.

Addition of Compound Denominations.

THE operation of adding numbers of compound denominations is essentially the same as in any other case, as will appear from a few examples.

Let it be required to add 50 hhd. 32 gal. 3 qt. 1 pt., 17 hhd. 27 gal. 1 qt., 22 hhd. 10 gal. 1 pt.

Since only things of the same kind are counted together, it becomes convenient to write the numbers expressing the same denomination in a vertical order, one under another, placing above each column the abbreviation of the denomination denoted by the numbers expressed below; thus:

hhd.	gal.	qt.	pt.
50	32	3	1
17	27	1	0
22	10	0	1
90	7	1	0

Adding first the numbers of the lowest denomination, which in this case is that of the pint, the sum is 2 pints.

As this is just equal to 1 quart, with nothing over, we write a zero underneath the column, and reserve 1 quart to be added with the numbers of quarts expressed

in the next column. The sum of the quarts is found to be 5, and reserving 1 gallon (that is, 4 quarts) there remains 1 quart, which is written underneath. Then adding the 1 gallon with the others the sum of the gallons is found to be 70, and as 63 gallons make a hogshead, there remain 7 gallons after reserving 1 hogshead to add with the others. Finally, having written the 7 gallons as a part of the result, the sum of hogsheads is found to be 90; and the entire sum is 90 hhd. 7 gals. 1 qt.

Again, let it be required to add 2 mi. 132 rd. 4 yd. 2 ft., 3 mi. 240 rd. 3 yd. 1 ft., 4 mi. 302 rd. 1 yd. 1 ft. Writing as before the numbers of the same denomination in a vertical order, we have

mi.	rd.	yd.	ft.	
2	132	4	2	
3	240	3	1	
4	302	1	1	
11	35	$3\frac{1}{2}$	1	
			1	6 in.
mi.	rd.	yd.	ft.	in.
11	35	3	2	6

In this case the foot is the denomination of smallest measure.

Adding first the feet, the sum is found to be 4 feet, or 1 yard and 1 foot, since 3 feet make a yard. Writing down 1 foot and counting 1 yard with the other yards, the sum is 9 yards, or 1 rod and $3\frac{1}{2}$ yards, since $5\frac{1}{2}$ yards make 1 rod.

Writing down $3\frac{1}{2}$ in the column of yards, 1 rod is counted with the other rods, making the sum 675 rods, or 2 miles 35 rods, since 320 rods make a mile. Writing down 35 under the column of rods, the 2 miles are counted with the others, making the sum 11 miles. Finally the $\frac{1}{2}$ yard (a part of $3\frac{1}{2}$ yards which appears in the result) may be expressed as 1 foot 6 inches, and 1 foot counted with the other, making 2 feet, so that the entire sum may be written 11 mi. 35 rd. 3 yd. 2 ft. 6 in.

In any case of the addition of numbers of compound denominations, *numbers of the same denomination are added together, and when this sum is enough to form one or more of the next higher denomination, such number of the next denomination is reserved to be added with others of the same kind, and the balance written down.*

Beginning with numbers of the lowest denomination, each kind is added in a similar way.

EXERCISES.

(1.)

	T.	lb.	oz.
	2	723	13
	4	842	10
	3	1175	11
Ans.	10	742	2

(2.)

lb.	oz.	pwt.	gr.
6	6	6	16
1	2	14	7
5	9	0	4
13	6	1	3

(3.)

A.	sq. rd.	sq. yd.	sq. ft.
50	50	20	6
10	144	16	7
25	87	25	4
39	1	0	8
125	124	$2\frac{1}{2}$	0

or

A.	sq. rd.	sq. yd.	sq. ft.	sq. in.
125	124	2	4	72

(4.)

cd.	cd. ft.	cu. ft.
5	5	5
8	7	6
3	6	12
4	2	8
22	5	15

(5.)

wk.	da.	hr.	m.	sec.
3	5	17	25	13
5	4	10	19	39
1	2	0	45	50
5	6	21	24	28

(6.)

12°	45′	55″
45	30	25
28	15	40
37	25	45

(7.)

cu. yd.	cu. ft.	cu. in.
10	20	844
12	12	1428
5	25	1264
2	18	675

(8.)

bu.	pk.	qt.	pt.
10	1	5	1
7	3	7	1
8	0	6	0
5	2	0	1

(9.)

rd.	yd.	ft.	in.
4	4	1	10
10	2	2	8
5	5	2	4

(10.)

mi.	fur.	rd.	yd.	ft.	in.
10	6	35	3	0	4
2	3	14	4	2	8
3	5	28	5	0	9

CHAPTER X.

Subtraction of Compound Denominations.

THIS will present little difficulty when the foregoing is understood. A few examples will serve to illustrate.

As in addition, it is found convenient to write numbers of the same denomination in a vertical order, and as in other cases of subtraction, the subtrahend is written under the minuend.

Let it be required to subtract 6 lb. 7 oz. 17 pwt. 18 gr. from 18 lb. 2 oz. 14 pwt. 22 gr. Writing these numbers in order we have

lb.	oz.	pwt.	gr.
18	2	14	22
6	7	17	18
11	6	17	4

Beginning with the lowest denomination, grains, the difference is easily found to be 4 grains. In the case of the pennyweights 17 cannot be taken from 14; that is, nothing can be added to 17 to make the sum 14, but 1 ounce may be taken from the number expressed in the next column, and reckoning it as 20 pwt., and, counting it with 14 pwt. first given, the sum is 34, from which 17 may be taken, leaving the difference 17 pwt.

Then 1 ounce remains in the minuend, and 7 ounces in the subtrahend. In this case we take 1 pound, or 12 ounces, from the 18 pounds, and add to 1 ounce, making 13 ounces, from which subtracting 7 ounces, the difference 6 ounces is found and written down. Finally, from the 17 pounds which remain in the minuend, subtracting 6 pounds, the difference is 11 pounds, which is written down, and the work is complete.

It is obvious that the principle of taking from a higher denomination is just the same as that already explained in the case of subtraction in Chap. V., Part I.

In the example just considered the minuend may be written as below, where the operation would appear more simple and the result be the same:

lb	oz.	pwt.	gr.
17	13	34	22
6	7	17	18
11	6	17	4

But with a little practice the method, as first explained, becomes easy to apply, and there will be no need to write the minuend in the second form.

From 10 mi. 2 fur. 15 rd. 0 yd. 2 ft. 4 in. subtract 2 mi. 7. fur. 39 rd. 5 yd. 2 ft. 10 in.

Writing these numbers in order we have:

	mi.	fur.	rd.	yd.	ft.	in.
	10	2	15	0	2	4
	2	7	39	5	2	10
	7	2	14	5	2	6
or	7	2	15	0	1	0

In this case it is found necessary to take 2 rods, or 11 yards, and then using 1 yard, or 3 feet, to count with the 2 feet, there remains 10 yards from which 5 yards may be subtracted, leaving the difference 5 as written.

It should be noticed that in the subtrahend there appears 5 yards 2 feet, which is more than $5\frac{1}{2}$ yards, and hence more than 1 rod.

This rod, counted with the 39 rods, makes 40 rods, or 1 furlong, and this again counted with 7 furlongs makes 8 furlongs, or 1 mile, and this with 2 miles makes 3 miles. So the subtrahend might be written 3 mi. 0 fur. 0 rd. 0 yd. $\frac{1}{2}$ ft. 10 in., or instead of the last two numbers, 1 ft. 4 in., and the subtraction might then be performed as follows:

mi.	fur.	rd.	yd.	ft.	in.
10	2	15	0	2	4
3	0	0	0	1	4
7	2	15	0	1	0

In the case of finding the difference between two dates, there is in practice an error which is usually neglected, but which should not be misunderstood.

Suppose it were required to find the interval between June 13, 1871, and August 4, 1872. The operation would usually be as follows:

1872	8	4
1871	6	13
1	1	21

and the result is 1 yr. 1 mo. 21 da.

In this case 30 days are reckoned to each month,

while, in fact, there are 31 days in one of the months between June 13th and August 4th.

In cases where greater accuracy is required, the number of days of each month is counted.

In all cases the operation of subtraction may be proved by adding subtrahend and difference together, and the sum should equal the minuend.

EXERCISES IN SUBTRACTION.

(1.)

wk.	da.	h.	min.	sec.
10	3	14	23	19
5	4	10	33	17
4	6	3	50	2

(2.)

T.	lb.	oz.
10	725	7
4	1245	12
5	1479	11

(3.)

lb.	oz.	pwt.	gr
24	6	6	6
10	8	8	8
13	9	17	22

(4.)

	mi.	fur.	rd.	yd.	ft.	in.
	8	0	0	0	0	0
	3	5	15	3	2	6
	4	2	24	$1\frac{1}{2}$	0	6
or	4	2	24	1	2	0

(5). From 10 da. 10 min. subtract 2 da. 2 h. 2 min. 2 sec. *Ans.* 7 da. 22 h. 7 min. 58 sec.

(6). From 50 gal. subtract 15 gal. 2 qt. 1 pt. 1 gi. *Ans.* 34 gal. 1 qt. 0 pt. 3 gi.

(7). What is the difference between 8 cd. 2 cd. ft. 4 cu. ft. and 3 cd. 5 cd. ft. 10 cu. ft.? *Ans.*

(8). What is the difference between 10 mi. 10 rd. and 2 mi. 2 fur. 2 rd. 2 yd. 2 ft. 2 in.? *Ans.*

(9). From 18 sq. yd. 2 sq. ft. 10 sq. in. subtract 3 sq. yd. 6 sq. ft. 100 sq. in. *Ans.*

(10). From 10 mi. 100 rd. 4 yd. 2 ft. subtract 275 rd 5 yd. 1 ft. 6 in.

(11). From 100° 20′ 20″ take 10° 45′ 45″. *Ans.*

(12). From 120° 55′ 15″ take 75° 34′ 38″ *Ans.*

In each of the foregoing examples the learner should prove the accuracy of the result by adding together subtrahend and difference.

CHAPTER XI.

Multiplication of Compound Denominations.

IN multiplying numbers of compound denominations the number of each denomination is multiplied separately, and the sum of the partial products constitutes the entire product, according to the general principle of multiplication.

Let it be required to multiply 7 gal. 2 qt. 1 pt. by 5.

Multiplicand and multiplier may be written in any convenient order—for instance, as follows:

gal.	qt.	pt.
7	2	1
		5
38	0	1

Beginning with 1 pint the product is 5 pints, equal to 2 quarts 1 pint. Writing 1 underneath, the number of quarts, 2 is reserved to be added to the product of quarts. This product is 10 quarts, to which adding 2 quarts the result is 12 quarts, or 3 gallons and nothing over. Writing 0 underneath the number of gallons, 3 is reserved to be counted with the product of gallons, which is 35. Adding 3 to this gives the result, 38 gallons, which is written underneath, and the work is complete.

In any other case the process is equally simple, and we may say in general, to multiply numbers of compound denominations, *multiply each denomination separately, beginning with the lowest, and when any partial product is sufficiently large to form one or more of the next higher denomination, such number of the next denomination is reserved to be added into the product of the next denomination, and the balance is written down.*

EXERCISES.

(1). Multiply 5 gal. 3 qt. 1 pt. by 8. *Ans.* 47 gal.

(2). Multiply 5 cd. 7 cd. ft. 10 cu. ft. by 6.
Ans. 35 cd. 5 cd. ft. 12 cu. ft.

(3). Multiply 17 mi. 4 fur. 25 rd. 12 ft. by 33.
Ans. 580 mi. 1 fur. 9 rd.

(4). Multiply 7 cu. yd. 18 cu. ft. 144 cu. in. by 12.
Ans. 92 cu. yd. 1 cu. ft.

(5). Multiply 8 oz. 5 pwt. 10 gr. by 18. *Ans.*

(6.) Multiply 5 h. 15 min. 15 sec. by 30. *Ans.*

(7). Multiply 2 sq. m. 21 A. 10 sq. rd. 50 sq. ft. by 8.
Ans.

(8). Multiply 6 T. 127 lb. 8 oz. by 16. *Ans.*

(9). Multiply 1 cong. 5 O. 6 f ℥. 6 f ʒ. 25 ♏. by 7.
Ans.

(10). Multiply 24 bu. 3 pk. 7 qt. 1 pt. by 64.
Ans. 1594 bu.

CHAPTER XII.

Division of Compound Denominations.

It will be easy to divide the numbers of compound denominations if we remember that division is the reverse of multiplication.

Let it be required to divide 38 gal. 0 qt. 1 pt. by 5.

The figures of dividend and divisor may be arranged as follows, and the quotient written below:

	gal.	qt.	pt.
5)	38	0	1
	7	2	1

Beginning with the highest denomination, gallons, the quotient of 7 divided into 38 gallons is 7 gallons, with a remainder 3 gallons. This remainder 3 gallons is equal to 12 quarts, and 5 divided into 12 quarts gives the quotient 2 quarts, with a remainder 2 quarts. This remainder 2 quarts is equal to 4 pints, which added to the 1 pint given in the dividend, make 5 pints. This divided by 5 gives the quotient 1 pint.

If the foregoing be compared with the process in the first example given as an illustration in Multiplication of numbers of compound denominations (page 53), it will be seen that one is the reverse of the other.

In any case, when the divisor is a single whole num-

ber, to divide numbers of compound denominations, divide the number of each denomination separately, beginning with the highest. If there be a remainder after the division of any denomination, reduce that remainder to the next lower denomination, and add it to the number of this lower denomination given in the dividend, using this sum as the dividend of that denomination. So continue till the lowest denomination is divided. Of course, if there be a remainder after the division of the lowest denomination, the quotient will be expressed fractionally.

If, in the last example, 38 gal. 1 pt. be expressed in pints, the result is 305 pints. This divided by 5 gives the quotient 61 pints, and this reduced to higher denominations gives 7 gal. 2 qt. 1 pt., the same as before.

This process may be used in all cases where the divisor is a single number.

It may happen that the divisor consists of numbers of compound denominations. In this case it will be necessary to reduce the divisor and the dividend to a single denomination of the same kind.

For example, to divide 10 mi. 50 rd. by 10 ft. 5 in. The divisor expressed in inches becomes 125 inches. The dividend expressed in inches becomes 634425 inches, and the quotient is easily found to be 5075.4.

EXERCISES.

(1). Divide 15 A. 3 R. 20 sq. rd. by 10.
Ans. 1 A. 2 R. 14 sq. rd.

(2). Divide 8 A. 60 sq. rd. by 15.
Ans. 2 R. $9\frac{1}{3}$ sq. rd.

(3). Divide 8 hhd. 42 gal. 2 qt. by 7. *Ans.*

(4). Divide 25 T. 1200 lb. by 8. *Ans.*

(5). Divide 184 lb. 6 oz. 12 pwt. 21 gr. by 24. *Ans.*

(6). Divide 15 A. 3 R. 20 sq. rd. by 1 A. 2 R. 14 sq. rd. *Ans.*

(7). Divide 8 hhd. 42 gal. 2 qt. by 1 hhd. 15 gal. *Ans.*

(8). Divide 25 T. 1200 lb. by 5 T. 200 lb. *Ans.*

(9). Divide 224 lb. 8 oz. 8 pwt. 20 gr. by 28 lb. *Ans.*

(10). Divide 1900 lb. 10 oz. 16 pwt. by 10 lb. 10 oz. 10 pwt. *Ans.*

COURSE IN ARITHMETIC.

PART III.

PRACTICAL APPLICATIONS.

PART III.

PRACTICAL APPLICATIONS.

CHAPTER I.

Introductory. Nature of Problems.

THE applications of the foregoing principles appear in the consideration of problems. A problem is a question, the answer to which becomes apparent through a course of reasoning called a solution.

Problems are met with in all departments of knowledge, but arithmetical problems require numerical processes in their solutions.

In some cases these solutions are quite simple, while again others require careful study. Sometimes a large class of problems may be solved by the same rule or method, but in many instances the student will need to use ingenuity, and rely upon sound judgment in seeking the best solution of problems.

A consideration of the several conditions upon which the solution of a problem depends, and of the processes of reasoning, is called an *analysis of a problem*. Usu-

ally a person makes an analysis in whole or in part as a preliminary to a solution, though when the solution is easy, one may scarcely be conscious of it.

In practice it is found that a formal statement of the analysis of a problem often serves to make the solution seem quite simple.

Problem.—If $\frac{4}{7}$ of an acre of land are worth \$25, what are $3\frac{1}{2}$ acres worth?

Analysis.—The value of a number of things may be found by multiplying the value of one thing by the number. Or the value of one thing may be found by dividing the value of a number by the number of things.

Hence, in the problem before us, we divide \$25 by $\frac{4}{7}$ to find the value of 1 acre, then multiply by $3\frac{1}{2}$ to find the value of $3\frac{1}{2}$ acres.

Solution.—$\$25 \div \frac{4}{7} = 25 \times \frac{7}{4} =$ value of 1 acre. $\$25 \times \frac{7}{4} \times 3\frac{1}{2} = \$25 \times \frac{7}{4} \times \frac{7}{2} = \$\frac{1225}{8} = \$153\frac{1}{8} =$ value of $3\frac{1}{2}$ acres.

Problem.—If $\frac{4}{9}$ of a barrel of flour cost \$8, what will $3\frac{1}{3}$ barrels cost?

The analysis is nearly the same as in the preceding problem, and need not be repeated.

Solution.—$\$8 \div \frac{4}{9} = \$3 \times \frac{9}{4} =$ cost of 1 barrel. $\$8 \times \frac{9}{4} \times \frac{10}{3} = \$60 =$ cost of $3\frac{1}{3}$, or $\frac{10}{3}$ barrel.

Problem.—How many bushels of corn, worth $\frac{5}{8}$ of a dollar a bushel, will pay for $\frac{2}{3}$ of a barrel of flour, at $\$8\frac{1}{3}$ a barrel?

Analysis.—The whole value of the flour, divided by

the value of 1 bushel of oats. would express the number of bushels required.

Solution.—$\$8\frac{1}{3} \times \frac{2}{3}$ = value of flour. $\$8\frac{1}{3} \times \frac{2}{3} \div \$\frac{5}{9} = \frac{25}{3} \times \frac{2}{3} \times \frac{9}{5} = 10$ *Ans.*

Problem.—If $\frac{3}{5}$ of a bushel of oats are worth $\frac{3}{8}$ of a bushel of corn, and corn be worth $\frac{2}{5}$ of a dollar a bushel, how many bushels of oats can be bought for $12?

Solution.—$\$\frac{2}{5} \times \frac{3}{8}$ = value of $\frac{3}{8}$ bushel of corn = value of $\frac{3}{5}$ bushel of oats.

Hence $\frac{2}{5} \times \frac{3}{8} \div \frac{3}{5} = \frac{2}{5} \times \frac{3}{8} \times \frac{5}{3} = \frac{1}{4}$ = value of 1 bushel of oats.

$\$12 \div \$\frac{1}{4} = 48$ = number of bushels sought.

PROBLEMS.

(1). A farmer wishes to buy an equal number of cows, sheep, and pigs, paying $22.25 for each cow, $3.50 for each sheep, and $1.75 for each pig. Having $110 to expend, what is the total number of animals he can purchase? *Ans.* 12.

(2). A dealer in lumber bought 12500 feet of lumber at $12.375 per M. and sold it at $1.625 per C. How much did he gain? *Ans.* $48.4375.

(NOTE.—Lumber merchants use M. and C., according to the Roman notation, in reckoning lumber.)

(3). At $37.50 per acre, how many acres of land may be bought for $4,706.25? *Ans.* $125\frac{1}{2}$.

(4). If $\frac{7}{8}$ yard of cloth cost $4.20, what will 5 yards cost? *Ans.* $24.00.

(5). A man bought a stack of hay at the rate of

$15.50 per ton, and it was found there were 4575 pounds of hay. What was the whole cost?

Ans. $35.45625.

(NOTE.—This would usually be reckoned as $35.46, since that expresses the value, to the nearest cent, the smallest denomination of money in use in this country. In what follows the answer will usually be given only to the nearest cent.)

(6). A lady bought $\frac{3}{4}$ yard velvet for $4.50. Another lady wished to buy $\frac{7}{8}$ yard at the same rate. What would it cost? *Ans.* $5.25.

(7). A merchant bought 12 pieces of muslin, each piece containing 48 yards, at $0.12½ per yard. What was the cost of the whole? *Ans.* $72.

(8). A farmer took 4 loads of hay to market. The loads (each with the wagon) weighed respectively 3,284 lb., 3,516 lb., 3,214 lb., and 2,812 lb., the wagon alone weighing 1,080 lb. At $14.75 per ton what did he receive for the hay? *Ans.* $62.81.

(9). John Brown received the following bill from his grocer:

LAWRENCE, KANSAS, July 4, 1876.

JOHN BROWN,

Bought of GEORGE FORD.

1876.			
Feb.	29	60 ℔. Soap....................@ 5½¢	$3.30
"	"	30 ℔. Starch....................@ 6½¢	1.95
"	"	15 ℔. Sugar....................@ 12½¢	1.87–5
March	13	18 ℔. Coffee....................@ 23 ¢	4.14
"	"	12 ℔. Cheese....................@ 14¾¢	1.77
May	31	25 ℔. Butter....................@ 31 ¢	7.25
			$20.28–5

Received Payment.

Upon examination an error was found. It is required to correct the error.

(10). It is required to verify the following bill:

St. Louis, Mo., March 15, 1877.

Moore & Bennet,

To J. Scomp & Co. *Dr.*

To 35 bbl. Flour........................	@ $4.75	$166.25
" 23 bbl. Flour........................	@ 5.25	120.75
" 150 lb. Dried Apples...	@ $12\frac{1}{2}$¢	18.75
" 15 bxs. Lemons......................	@ $8.25	123.75
" 7 bxs. Raisins......................	@ 4.85	33.95
		$463.45

Received Payment,

J. Scomp & Co.

(11). It is required to make a bill of the following named items:

A. F. Bates & Co. bought of Broker, Steele & Co., Kansas City, August 20, 1877, 4 doz. inkstands, at $1.87½; 12 boxes steel pens, at $1.12½; 10 reams note-paper, at $3.25; 6 reams legal cap, at $3.50; 5 doz. blank-books, at $1.45.

(12). It is required to make a bill of the following named items:

John Smith bought of Ridenour & Baker 6 lb. chocolate, at 17 cts.; 25 lb. flour. at 6 cts.; 3 pairs boots, at $4.50; 5 lb. tea at 85 cts., 3 boxes raisins at $3.25.

CHAPTER II.

Problems of Ratio and Proportion.

SECTION I.

Ratio.

IN comparing one number with another, or in comparing one quantity with another, we may state the difference between the two, or we may state how many times one is greater than another. Thus we may say 12 is 4 more than 8, or we may say 12 is $1\frac{1}{2}$ times 8.

In this chapter we shall consider the relation expressed by the quotient of one number divided by another, or of one quantity divided by another. The quotient expressing the relation of one number or one quantity to another number or quantity, is called *ratio.*

The ratio may be indicated by the fractional sign of division, or by a colon placed between the figures of the numbers. Thus the ratio of 2 to 3 may be indicated by $\frac{2}{3}$, or 2 : 3. Some writers place the terms in an inverse order; that is, would write 3 : 2 to express $\frac{2}{3}$, but usage seems to favor the order first stated.

It is probable that the symbol of the colon is derived from the ordinary sign of division by omitting the line from between the dots.

The first term of an indicated ratio is called the *antecedent*, the second term is called the *consequent.*

Thus in the ratio $\frac{2}{3}$, or 2 : 3, 2 is the antecedent and 3 the consequent; and again, in the case of $\frac{8}{6}$, or 8 : 6, 8 is the antecedent and 6 is the consequent.

The two numbers together are often called a *couplet*.

As in division so in an indicated ratio, the two numbers must express things of the same kind, or else, in forming a ratio, they must be regarded merely as numbers of things. Illustrations of this precept will occur as we proceed.

From the foregoing it seems that the antecedent, the consequent, and the ratio are in effect in the relation to each other of dividend, divisor, and quotient.

SECTION II.

Proportion.

An expression of two equal ratios is called a proportion. It is indicated by placing a double colon, or a sign of equality, between the symbols of the ratios. Thus 2 : 3 : : 4 : 6, or $\frac{2}{3} = \frac{4}{6}$, expresses a proportion, and the numbers themselves are said to be *in proportion*, or they *form a proportion*.

The above expressions are read, "2 is to 3 as 4 is to 6," or "the ratio of 2 to 3 equals the ratio of 4 to 6," or "2 divided by 3 equals 4 divided by 6."

The four numbers which form a proportion are called the terms of the proportion. The first and last are called the *extremes*, the second and third are called the *means*, and it is easily shown that the product of the extremes is equal to the product of the means.

Thus, in the proportion $\frac{2}{3} = \frac{4}{6}$, if the ratio expressed by each couplet be multiplied by the product of the consequents (or divisors), that is, by 3×6, the result would be $\frac{2}{3} \times 3 \times 6 = \frac{4}{6} \times 3 \times 6$, or $2 \times 6 = 4 \times 3$. And in any other case, multiplying each of the indicated ratios of a proportion by the product of the consequents, would cancel the consequents and leave the product of the means equal to the product of the extremes.

(But these products must be understood as of *numbers* of things.)

This principle is conveniently applied in the solution of many problems in which the number sought forms the fourth term of a proportion, of which three are known. If we have three terms given—for example, $2 : 3 :: 4 :$ ——, the last not known—this can be found, for the product of the means, $3 \times 4 = 12$, equals the product of the extremes, of which one is known to be 2, and the other must be such that, multiplied by 2, will produce 12; that is, it must be $\frac{12}{2} = 6$. In any case when the fourth term of a proportion is unknown but required, the product of the second and third may be divided by the first. Or if the third term is wanting, as $2 : 3 ::$ — $: 6$, it may be found by multiplying together the first and fourth and dividing by the second.

The method of solving problems by means of finding the fourth term of a proportion of which three are given, is called the Rule of Three.

Problem.—If a pole 5 feet in height has a shadow $7\frac{1}{2}$ feet long, how high is a pole, the shadow of which, at the same time, is 20 feet long?

Here are given the height of one pole and the length of two shadows, and it is evident that the ratio of the lengths of the shadows must equal the ratio of the heights of the poles, considered in the same order.

The required pole is longer than the given pole, since its shadow is longer, so we say, "As the length of the shadow of the given pole is to the length of shadow of the required pole, so is length of given pole to length of required pole." That is, $7\frac{1}{2} : 20 :: 5 : —$, or $\frac{20 \times 5}{7\frac{1}{2}} = 13\frac{1}{3}$. *Ans.*

In the case of any problem which is to be solved by the rule of three, determine by inspection which of the given numbers is compared directly with the number sought, and use that for the third term. Of the two remaining numbers use the smaller for the first or second term, according as the number sought is to be greater or less than the third.

By practice the application of this rule becomes easy in the case of any problem solvable by this method.

The problem which follows will serve for further illustration.

Problem.—A farmer meeting another says: "I have sold 25 pigs and 40 sheep." The other replies: "I have sold 24 horses, and a number of cows which forms the same ratio with the number of horses, as the ratio of the number of pigs to the number of sheep which you have sold." Required the number of cows.

As the number of horses is compared with the number (of cows) sought, it is therefore selected for the third term.

An inspection of the statement shows the number of cows is less than the number of horses as the number of pigs is less than the number of sheep. Hence the larger of the two remaining numbers is selected for the first, and we have three terms of the proportion.

$$40 : 25 :: 24 : \text{—}, \text{ or } \tfrac{24 \times 25}{40} = 15. \quad \textit{Ans.}$$

It should be remarked that the ratios here are not between pigs and sheep, cows and horses, but between numbers of things. Thus it is not $\frac{40 \text{ sheep}}{25 \text{ pigs}}$, but $\frac{40 \text{ things}}{25 \text{ things}}$, or $\frac{40 \text{ animals}}{25 \text{ animals}}$. And in general, in comparing numbers of different kinds of things, the ratio will be understood merely as between individual things.

PROBLEMS.

(1). If 3 men can do a piece of work in 10 days, how long will it require 6 men to do the same?

Ans. 5 days.

(2). If 24 men can do a piece of work in 10 days, how long will it take 5 men to do the same?

Ans. 48 days.

(3). If 2 bushels of corn are worth 3 bushels of oats, how many bushels of corn are worth 28 bushels of oats? *Ans.* $18\frac{2}{3}$ bushels.

(4). A man makes a profit of $1500 in 4 months. At the same rate what profit will he make in 10 months? *Ans.* $3750.

(5). A man gains $193 on goods that cost $650. At

the same rate what value of goods must he buy in order to make a profit of \$1000? *Ans.* \$3367.87.

(6). If $\frac{3}{4}$ bushel of peaches cost \$0.37½, what part of a bushel can be bought for \$0.30? *Ans.*

(7). If \$800 be the annual profit of a farm worth \$3500, what, at the same rate, should be the annual profit of a farm worth \$6500? *Ans.*

(8). If 15 head of cattle require 18 A. 3 R. of pasture, it is required to find how much pasture will be needed for 50 head of cattle. *Ans.*

SECTION III.

Compound Proportion.

The expression of the product of two or more ratios equal to a single ratio, is called a *compound proportion.*

Thus $\frac{2}{3} \times \frac{5}{6} = \frac{25}{45}$, or $\left.\begin{matrix} 2 : 3 \\ 5 : 6 \end{matrix}\right\} :: 25 : 45$, is a compound proportion. There are many problems whose solutions may be obtained by finding the last term of a compound proportion of which the other terms are given.

Problem.—If 10 men, working 12 hours a day, perform a task in 24 days, in how many days can 18 men perform an equal task, working 8 hours a day?

It is evident from the statement that the number of days sought is to be compared with the 24 days named, and hence 24 will be selected for the third term. Again, in the task done, 10 men were employed, while in the task to be done 18 men are to be employed. Hence the time sought will be less on this account, and

the greater number 18 is made the first term of the ratio 18 : 10.

Further, working only 8 hours a day more days will be required, and hence, in the ratio of hours, the smaller number is used for the first, that is, 8 : 12.

Arranging the terms in order, we have

$$\left.\begin{array}{r}18:10\\8:12\end{array}\right\}::24:\text{—},\text{ or }\frac{24\times10\times12}{18\times8}=20\text{ days.}\quad \textit{Ans.}$$

In general, in the solution of any problem to which the method of compound proportion is applicable, select that number for the third term which is to be directly compared with the number sought. Of the remaining numbers form couplets of those which are to be compared together, as in the method of simple proportion. Then multiply the product of the consequents by the number selected for the third term, and divide this result by the product of the remaining antecedents.

PROBLEMS.

(1). If it requires 10 horses to plough 8 hectares in 4 days, how many horses will be required to plough 20 hectares in 10 days? *Ans.*

(2). If a cistern 4 meters long, 2½ meters wide, and 3 meters deep holds 410 barrels, what will be the capacity of a cistern which is 5 meters long, 2 meters wide, and 6 meters deep? *Ans.*

(3). If there are 36 steres of wood in a pile 15 meters long, 1 meter wide, and 2.4 meters high, how many steres of wood will there be in a pile 20 meters long, 4 meters wide, and 3.6 meters high? *Ans.* 288.

CHAPTER III.

Involution.

THE process of finding any power of a number is called *involution.*

As already explained, the second, third, fourth, or any higher power of a number, may always be obtained by simple multiplication. Thus the third power of 4 is $4 \times 4 \times 4 = 64$, the fourth power of $\frac{2}{3}$ is $\frac{2}{3} \times \frac{2}{3} \times \frac{2}{3} \times \frac{2}{3} = \frac{16}{81}$, and the fifth power of .02 is $.02 \times .02 \times .02 \times .02 \times .02 = .0000000032$.

In the remaining part of this chapter we shall consider only the second and third powers of numbers.

It is worth noticing that the square of any whole number expressed in the Arabic notation requires twice as many figures, or one less than twice as many, as the figures of the given number.

This is easily seen by inspecting the list that follows:

$$\left\{\begin{array}{ll} 1^2 & = 1. \\ 9^2 & = 81. \end{array}\right.$$

$$\left\{\begin{array}{ll} 10^2 & = 100. \\ 99^2 & = 9,801. \end{array}\right.$$

$$\left\{\begin{array}{ll} 100^2 & = 10,000. \\ 999^2 & = 998,001. \end{array}\right.$$

$$\left\{\begin{array}{ll} 1000^2 & = 1,000,000. \\ \&c. & \end{array}\right.$$

The squares of the greatest and least numbers expressed by first one figure, then by two figures, then by three figures, and so on, are seen to conform to the principle above stated.

In a similar way the figures of the cube of any whole number are three times as many as those of the given number, or lack not more than two of that number.

Thus

$$\begin{aligned} 10^3 &= 1000. \\ 100^3 &= 1,000,000. \\ 1000^3 &= 1,000,000,000. \end{aligned}$$

It is evident that the cube of 99 will be less than the cube of 100, and will, therefore, require 6 figures; and in a similar way the cube of 999 will require 9 figures, and so on, according to the principle stated.

It will also be useful to trace the effect of each of two parts of a number in forming the square of that number.

For example, in squaring 54. Multiplying and retaining each partial product distinctly, we have

$$\begin{array}{r} 54 \\ 54 \\ \hline 16 \\ 200 \\ 200 \\ 2500 \\ \hline 2916 \end{array} \qquad \text{or} \qquad \begin{array}{r} 50+4 \\ 50+4 \\ \hline 200+16 \\ 2500+200 \\ \hline 2500+2\times200+16 \end{array}$$

and this result may again be written

$$50^2+2\times4\times50+4^2.$$

That is, we have in the square of this number, composed of tens and units, the square of the tens, + twice the product of the tens × units + the square of the units. An inspection of the operation indicated above will make it evident that a similar result would take place in squaring any other number composed of tens and units. Indeed a similar result would take place if a number were separated into any other two parts. Suppose 7 be taken as the sum of 4+3, and find the square by multiplying the parts separately.

$$\begin{array}{r} 4+3 \\ 4+3 \\ \hline 12+9 \\ 16+12 \\ \hline \end{array}$$

$$16+2\times 12+9=4^2+2\times 3\times 4+3^2=49=7^2.$$

And here again we find the

$$(\text{1st part})^2+2\times\text{1st part}\times\text{2d part}+(\text{2d part})^2.$$

Suppose we find the cube of 54 by multiplying the separate parts of the square. First we may write

$$(54)^2=50^2+2\times 4\times 50+4^2$$

and indicating the multiplication of the parts,

$$
\begin{array}{r}
50^2 + 2\times4\times50 + 4^2 \\
50 + 4 \\
\hline
4\times50^2 + 2\times4^2\times50 + 4^3 \\
50^3 + 2\times4\times50^2 + 4^2+50 \quad \\
\hline
\end{array}
$$

have $50^3 + 3\times4\times50^2 + 3\times4^2 + 50+4^3 =$

$$
\begin{array}{r}
125000 \\
30000 \\
2400 \\
64 \\
\hline
157464
\end{array}
$$

the same as obtained by the usual method of direct multiplication.

Thus the cube of any number composed of tens and units contains $(\text{tens})^3 + 3\ (\text{tens})^2 \times$ units $+ 3$ tens $\times$ $(\text{units})^2 + (\text{units})^3$, since the cube of any number may be found in a similar way by multiplying the parts separately. And it is also clear that were a number separated into any two parts a similar statement would be true.

For illustration, $7^3 = 7\times7\times7 = 343$.

But $7 = 3 + 4$, and

$$
\begin{array}{r}
7^2 = 3^2 + 2\times3\times4 + 4^2 \\
3 + 4 \\
\hline
3^2\times4 + 2\times3\times4^2 + 4^3 \\
3^3 + 2\times3^2\times4+3\times4^2 \quad \\
\hline
\end{array}
$$

multiplying again by $3+4$ we have, $3^3+3\times3^2\times4+3\times3\times4^2+4^3.$

In adding the partial products above indicated it is evident that $2\times3\times4^2$, added to 3×4^2, gives $3\times3\times4^2$,

since 2 times and (1 time) once make 3 times, and similarly in adding $3^2 \times 4$, and $2 \times 3^2 \times 4$, the sum is $3 \times 3^2 \times 4$.

But

$$
\begin{aligned}
3^3 &= 27 \\
3 \times 3^2 \times 4 &= 108 \\
3 \times 3 \times 4^2 &= 144 \\
3^3 &= 27 \\
\hline
\text{entire sum} &= 343
\end{aligned}
$$

Note.—It is scarcely necessary to remark, that the square of a fractional number is obtained by squaring both numerator and denominator.

EXERCISES.

(1). Find the square of 36 first by direct multiplication, then as the sum of 3 tens and 6 units.

(2). Find the value of $(83)^2$ and $(80+3)^2$.

(3). Find $(16)^2$ and $(7+9)^2$.

(4). Find value of $(100+4)^2$.

(5). Find value of $(25+3)^2$.

(6). Find value of $(90+1)^3$.

(7). Find value of $(60+3)^3$.

(8). Find value of $(9+7)^3$.

(9). Find value of $(100+1)^3$.

(10). Find value of $(1000+1)^3$.

(11). Find value of $(20+40)^3$.

(12). Find value of $(8+6)^3$.

CHAPTER IV.

Evolution of Roots.

The number used as a factor to produce a given power is called the root of that power, and the root is named the 2d, or 3d, and so on, according to the degree of the power. Thus if any number is the 3d power of a root, the root is called the 3d or cube root of that power, and in a similar way the 4th root, and so on.

The process of finding the root of a given power is called *evolution*, but in this treatise we shall only consider the evolution of the 2d and 3d roots, called respectively the *square root* and the *cube root*.

SECTION I.

The Square Root.

In finding the square root of any number it is only necessary to retrace the method used to find the square of the sum of two numbers. Some larger portion of the root is first estimated by inspection and the balance found by trial.

Suppose it were required to find the square root of 4096. The figures of the square root will not exceed two; that is, the root will be less than 100. It is easy

to find that the required root is more than 60 and less than 70, because $60^2 = 3600$ is less, and $70^2 = 4900$ is more than the given number. Then we may consider 60, or 6 tens, as the tens part of the root. But in the last chapter it was found that the square of a number of tens and units consists of three parts: 1°, the square of the tens; 2°, $2 \times$ tens $\times$ units; 3°, square of units. If the first part be taken away the other two will remain, and it is obvious that the second part, that is, $2 \times$ tens $\times$ units, is much the larger of the two portions.

Then from 4096 subtract $60^2 = 3600$ and there remains 496.

$$
\begin{array}{lr}
\text{Thus} & 4096 \\
 & 3600 \\
\hline
\text{remainder} = & 496
\end{array}
$$

Since this remainder is chiefly composed of $2 \times 60 \times$ units $= 120 \times$ units, we shall find the number of units very nearly by dividing by 120; that is, by $2 \times$ tens.

Dividing 496 by 120, the integral part of the quotient is 4. Consider 4 as the number of units in the required root, and find by trial whether it be the exact part required.

$$
\begin{array}{lr}
\text{We have } 2 \times \text{tens} \times \text{units} = 2 \times 60 \times 4 = & 480, \text{ and} \\
\text{Square of units} = 4 \times 4 = & 16, \text{ and} \\
\hline
 & 496
\end{array}
$$

thus find that 4 fulfills the condition.

The different parts of the operation may then be indicated more briefly as follows:

$$
\begin{array}{rrl}
 & 4096\,(& 60 = \text{1st part of root.} \\
\text{Subtract } 60^2 = & \underline{3600} & 4 = \text{2d part of root.} \\
\text{Trial divisor} = 2 \times 60 = 120\,) & \underline{496} & = \text{1st remainder.} \\
2 \times 60 \times 4 = & 480 & \\
4 \times 4 = & \underline{16} & \\
 & \underline{\underline{496}} & \\
 & 0 & = \text{2d remainder.}
\end{array}
$$

Again, to find the square root of 1444. Proceeding as before, we have

$$
\begin{array}{rrl}
 & 1444 & \underline{|\,30 = \text{1st part.}} \\
 & \underline{900} & \;\;9 = \text{2d part.} \\
2 \times 30 = 60\,) & \underline{544} & \\
2 \times 30 \times 9 = & 540 & \\
9^2 = & \underline{81} & \\
 & 621 &
\end{array}
$$

In this case, using 2×30, or 60, as the trial divisor, we obtain 9 as the second part of the root. But on completing the square, find that this is too large a number. Therefore trying the next smaller number, 8, the result is as follows:

$$
\begin{array}{rrl}
 & 1444 & \underline{|\,30 + 8 = 38} \\
 & \underline{900} & \\
2 \times 30 = 60\,) & \underline{544} & = \text{1st remainder.} \\
2 \times 8 \times 30 = & 480 & \\
8^2 = & \underline{64} & \\
 & \underline{544} & \\
 & 0 & = \text{2d remainder.}
\end{array}
$$

It may happen that the given number has no exact square root.

Let it be required to find the square root of 500.

$$
\begin{array}{rr|l}
 & 500 & \underline{20+2} \\
 & 400 & \\
\hline
2\times 20 = 40\,) & 100 & \\
\hline
2\times 2\times 20 = & 80 & \\
2^2 = & 4 & \\
\hline
 & 84 &
\end{array}
$$

The second part of the root is evidently more than 2 and less than 3, and we may find the number of decimal tenths in the root by regarding 2 as a part of the root, and proceeding as in the first instance.

$$
\begin{array}{rr|l}
 & 500 & \underline{20+2+.3+.06 = 22.36.} \\
 & 400 & \\
\hline
2\times 20 = 40\,(& 100 & = \text{1st remainder.} \\
\hline
2\times 2\times 20 = & 80 & \\
2^2 = & 4 & \\
\hline
 & 84 & \\
\hline\hline
2\times 22 = 44\,) & 16 & = \text{2d remainder.} \\
\hline
2\times .3\times 22 = & 13.2 & \\
.3^2 = & .09 & \\
\hline
 & 13.29 & \\
\hline\hline
2\times 22.3 = 44.6) & 2.71 & = \text{3d remainder} \\
\hline
2\times 22.3\times .06 = & 2.676 & \\
.06^2 = & .0036 & \\
\hline
 & 2.6796 & \\
\hline\hline
 & .0304 & = \text{4th remainder.}
\end{array}
$$

Adding together the partial roots found so far, we have 22.36, with a remainder, and the process could be carried on indefinitely without ever reaching the exact root, for the square of the lowest decimal would not be zero, while all the decimal orders of the given number are zero, and hence there must always be a remainder.

Let it be required to find the square root of 54756. We first find by trial the highest number of hundreds whose square is less than the given number. This is at once found to be 200, and we then proceed as follows:

$$
\begin{array}{rr|l}
 & 54756 & 200+30+4 \\
 & 40000 & \\
\cline{2-2}
2\times 200 = 400) & 14756 & = \text{1st remainder.} \\
\cline{2-2}
2\times 200\times 30 = & 12000 & \\
30^2 = & 900 & \\
\cline{2-2}
 & 12900 & \\
\cline{2-2}
2\times 230 = 460) & 1856 & = \text{2d remainder.} \\
\cline{2-2}
2\times 230\times 4 = & 1840 & \\
4^2 = & 16 & \\
\cline{2-2}
 & 1856 & \\
\cline{2-2}
 & 0 & = \text{3d remainder.}
\end{array}
$$

It should be observed in this case that in the first subtraction we subtract the square of 200, in the first two subtractions together we subtract the square of 230, and in the three subtractions the square of 234 is subtracted, and nothing remains.

From the several preceding illustrations it will be easy to deduce the following

Rule for Extracting the Square Root.

Find by trial the highest number of the highest order the square of which is less than the given number, and regarding this as a part of the root, subtract its square from the given number.

Use twice the first part of the root as a trial divisor, and find, as a second part of the root, the highest number of the next lower order, such that the sum of its square, and twice the product of the two parts, shall not exceed the first remainder.

If there be a second remainder in subtracting this sum from the first remainder, use the sum of the two parts of the root found as one part, and proceed as before to find the highest number of the next lower order in the root, and so continue until there is no remainder, or until the root is extended to as low an order of decimals as may be desired.

The symbol used to indicate that the square root is required is formed as follows, $\sqrt{\ }$, and is placed above the figures of the number whose square root is required, and is called the "radical sign."

The form of this sign was derived from the use of the initial letter of the word *root*, $r\overline{\ \ }$, followed by a straight line.

The indicated root of a number which cannot be exactly expressed is often called a "surd."

The roots of a higher order are indicated by using the same symbol, with the addition of a figure called

an index, to show the order of the root. Thus $\sqrt[3]{8}$ means the cube root of 8.

The square root is always understood when no index is written.

EXERCISES.

(1). Show $\sqrt{12321} = 111$.

(2). $\sqrt{16499844} = 4062$.

(3). $\sqrt{\frac{4}{9}} = \frac{2}{3}$.

(4). $\sqrt{\frac{25}{1024}} = \frac{5}{32}$.

(5). $\sqrt{1000} = 31.4+$

(6). $\sqrt{6205081} = 2491$.

(7). $\sqrt{.0081} = .09$.

(8). $\sqrt{.451584} = .672$.

(9). $\sqrt{\frac{6}{7}} = \sqrt{\frac{42}{49}} = \frac{1}{7}\sqrt{42.} = .92582+$

(10). Find the value of $\sqrt{\frac{1}{5}}$.

(11). $\sqrt{.049}$, $\sqrt{.144}$, $\sqrt{2.89}$.

SECTION II.

Cube Root.

Remembering that the cube of any number which is composed of tens and units consists of the sum of $(\text{tens})^3 + 3 \times (\text{tens})^2 \times (\text{units}) + 3 \times \text{tens} \times (\text{units})^2 + (\text{units})^3$, it will be easy to find the cube root of any given number by retracing the steps thus indicated in forming the cube.

For illustration, let it be required to extract the cube root of 373248.

The required root is more than 10 and less than 100, and by successive trial it is found to be more than 70 and less than 80.

Consider 70 as one part of the root, subtract its cube, and proceed as follows:

$$
\begin{array}{lrl}
 & 373248 & \underline{|70+2} \\
 & \underline{343000} & \\
3\times 70^2 = 14700 &)\,30248 & = \text{1st remainder.} \\
3\times 70^2\times 2 = & 29400 & \\
3\times 70\times 2^2 = & 840 & \\
2^3 = & \underline{8} & \\
 & 30248 &
\end{array}
$$

Here we use 3 times the square of the first part of the root as a trial divisor to find the second part, because it is evidently the largest factor in the remainder.

Having obtained 2 as the second part of the root, we proceed to ascertain, by completing the cube, whether it is the exact root sought. By this means it is found to fulfil the exact conditions, and hence 72 is the complete root sought.

Again, to find the cube root of 160,103,007. Nine figures of the given number show that the cube root will be expressed by three figures and that the highest order will be hundreds. A little further examination shows the root sought is more than 500 and less than 600. Using 500 as one part of the root, the work that follows will be easily understood.

$$
\begin{array}{rrl}
 & 160103007 & \underline{| \ 500+40+3} \\
500^3 = & 125000000 & \\
\hline
3\times 500^2 = 750000\,) & 35103007 & = \text{1st remainder.} \\
3\times 500^2\times 40 = & 30000000 & \\
3\times 500\times 40^2 = & 2400000 & \\
40^3 = & 64000 & \\
\hline
 & 32464000 & \\
\hline
3\times 540^2 = \ 874800\,) & 2639007 & = \text{2d remainder.} \\
3\times 540^2\times 3 = & 2624400 & \\
3\times 540\times 3^2 = & 14580 & \\
3^3 = & 27 & \\
\hline
 & 2639007 & \\
\hline
 & 0 & = \text{3d remainder.}
\end{array}
$$

A careful examination will show that this is merely a retracing of the method of forming the cube of the sum of two numbers.

From the foregoing it will be easy to deduce the following

Rule for Finding the Cube Root of any Number.

Find by trial the highest number of the highest order, the cube of which is contained in the given number, and regarding it as a first part of the root sought, subtract its cube from the given number.

Divide the remainder by 3 times the square of the first part found, as a trial divisor, and by this means find the highest number of the next lower order, such that its cube, added to 3 times its square multiplied into the first part of the root, plus 3 times the product

of this number multiplied into the square of the first part, shall not exceed the first remainder.

Subtract this sum from the first remainder, and if there be a second remainder, use the sum of the two parts of the root found as one part, and proceed as before to find the highest number of the next lower order that may be contained in the root, and so continue until there is no remainder, or until the root is ascertained to a sufficiently low order of decimals.

To illustrate the case of a decimal root which can never be exactly found, let it be required to find the cube root of 10000.

$$
\begin{array}{rrl}
 & 10000 & \underline{|\ 20+1+.5+.04} \\
 & 8000 & \\
\hline
3\times 20^2 = 1200\,) & 2000 & = \text{1st remainder.} \\
3\times 20^2\times 1 = & 1200 & \\
3\times 20\times 1^2 = & 60 & \\
1^3 = & 1 & \\
\hline
 & 1261 & \\
3\times 21^2 = 1323\,) & 739 & = \text{2d remainder.} \\
3\times 21^2\times .5 = & 661.5 & \\
3\times 21\times .5^2 = & 15.75 & \\
.5^3 = & .125 & \\
\hline
 & 677.375 & \\
3\times 21.5^2 = 1386.75\,) & 61.625 & = \text{3d remainder.} \\
3\times 21.5^2\times .04 = & 55.4700 & \\
3\times 21.5\times .04^2 = & .10320 & \\
.04^3 = & .000064 & \\
\hline
 & 55.573264 & \\
 & 6.051736 & = \text{4th rem.}
\end{array}
$$

So far, then, the root is found to be 21.54 +, with a remainder from which lower orders of decimals could be found without end, and the exact root never be obtained.

The symbol of the cube root has been previously explained, and is formed by placing the index 3 at the left and above the radical sign under which the figures of the number are written.

In finding the cube root of a fractional number we take the cube root of the numerator and the cube root of the denominator, because each is cubed in cubing a fractional number. Thus $\sqrt[3]{\frac{8}{27}} = \frac{2}{3}$, because $(\frac{2}{3})^3 = \frac{8}{27}$, and similarly $\sqrt[3]{\frac{64}{343}} = \frac{4}{7}$.

If it happen that the denominator is an exact cube, or has an exact cube root, while the numerator has not, the cube root of the numerator can be found approximately in decimals, and this result divided by the cube root of the denominator. Thus the cube root of $\frac{10}{27}$ is $\sqrt[3]{\frac{10}{27}} = \frac{1}{3}\sqrt[3]{10} = \frac{2.154+}{3} = .718 +$.

If it happen that the denominator is not an exact cube, then both terms of the fractional number may be multiplied by some number which will change the denominator to a perfect cube, and then proceed as in the last case. For example, to find the cube root of $\frac{4}{9}$ we have $\sqrt[3]{\frac{4}{9}} = \sqrt[3]{\frac{12}{27}} = \frac{1}{3}\sqrt[3]{12}$, and the operation may be completed as in the preceding example.

Of course the cube root of numerator and of denominator in the given form could be obtained approximately in decimals, but it would require much more labor than the method just explained.

EXERCISES.

(1). $\sqrt[3]{13824} = 24.$

(2). $\sqrt[3]{42875}.$

(3). $\sqrt[3]{274625}.$

(4). $\sqrt[3]{3048625}.$

(5). $\sqrt[3]{15625}.$

(6). $\sqrt[3]{80}.$

(7). $\sqrt[3]{\frac{6}{7}}.$

(8). $\sqrt[3]{\frac{72}{243}}.$

(9). $\sqrt[3]{20}$, to the 4th decimal place.

(10). $\sqrt[3]{25}.$

(11). $\sqrt[3]{1.25}.$

(12). $\sqrt[3]{3.375}.$

(13). $\sqrt[3]{.00125}.$

(14). $\sqrt[3]{4096}.$

(15). $\sqrt[3]{1157.625}.$

CHAPTER V.

Percentage.

SECTION I.

First Principles.

PERCENTAGE is an allowance by the hundred. The custom of reckoning by the hundred is of ancient origin, and at the present time prevails in a large share of the transactions of ordinary business.

Per cent is derived from the Latin *per centum*, meaning by the hundred. *Rate per cent* means simply *rate by the hundred. Six per cent* means *six of every hundred.*

If it be said that in a school of 180 scholars 5 per cent are absent, it is meant that $\frac{5}{100}$ of the whole number are absent; that is, $\frac{5}{100} \times 180 = 9$ are absent.

Or if it be said that a man receiving an inheritance of \$2000 wasted 20 per cent in one year, it is understood that he wasted $\$2000 \times \frac{20}{100} = \$400$.

In the first case it is said the *percentage* of absentees, reckoned at 5 per cent, is 9; in the latter case the percentage of \$2000, reckoned at 20 per cent, is \$400.

The rate per cent, divided by 100, expressed the rate per unit. Thus, if the rate per cent is 5, the rate per

unit is $\frac{5}{100}$, or .05, and when the rate per cent is 20, the rate per unit is $\frac{20}{100}$, or .20, or simply .2.

If the rate per cent is $\frac{1}{2}$ or .5, the rate per unit is .005. Again, for a rate per unit .004 the rate per cent is $\frac{4}{10}$, for a rate per unit of .084$\frac{2}{7}$ the rate per cent is 8.4$\frac{2}{7}$, or 8$\frac{3}{7}$, and $\frac{5}{11}$ per cent is equivalent to .004$\frac{6}{11}$ per unit.

In business transactions the symbol % is used for the words per cent, and accordingly 6 per cent is expressed 6%, and 2$\frac{1}{2}$ per cent 2$\frac{1}{2}$%.

Problem.—A drover started for market with 600 sheep, but on the way lost 4% of them. How many were lost and how many remained? As 4% is equivalent to $\frac{4}{100}$ of any number, then $\frac{4}{100}$ of $600 = 24$, the number lost, and $600 - 24 = 576$, the number remaining.

In this, as in the previous examples, the percentage is found by multiplying the number (of which the percentage is required) by the rate per unit.

It may be remarked that 100% is $\frac{100}{100}$ of a number, or the number itself; 200% is twice a number; 300% three times, and so on.

Problem.—A man has a salary of $2400 per year; he pays 12% for board, 8% for clothing, 10% for books and periodicals, 10% for miscellaneous expenses; the balance is deposited in bank. How much is thus deposited.

Solution.—12% + 8% + 10% + 10% = 40%. $2400 × .40 = 960. $2400 − $960 = $1440. *Ans.*

PROBLEMS.

(1). A man had \$5000 in bank, but drew out first 20%, and then 25% of what remained. Afterwards he replaced 20% of all he had drawn out; how much then remained in bank? *Ans.* \$3400.

(2). Find $16\frac{2}{3}$% of 240 sheep. *Ans.* 40 sheep.

(3). Find 18% of 15 gallons. *Ans.* $2\frac{7}{10}$ gallons.

(4). Find $4\frac{1}{2}$% of \$7.50. *Ans.* \$0.$33\frac{3}{4}$.

(5). Find $6\frac{1}{4}$% of \$18.

(6). Find $8\frac{1}{3}$% of \$24.36.

(7). Find 75% of \$0.13.

(8). Find 180% of \$25.

(9). Find 500% of \$1000.

(10). Find .5% of \$1300.

(11). Find $\frac{1}{8}$% of \$1000.

It is often required to find the rate per cent when the percentage is known. Thus, a man had \$48, but lost \$4; it is required to find the rate per cent of his loss.

Since the product of a number multiplied by the rate per unit gives the percentage, the quotient of the percentage divided by the number will give the rate per unit. (That is, a product divided by one of two factors must give a quotient equal to the other factor, which in this case is the rate per unit.) The rate per unit, multiplied by 100, gives the rate per cent. Hence $\frac{4}{48}$ = rate per unit, and $\frac{4}{48} \times 100 = 8\frac{1}{3}$, the rate per cent.

PROBLEMS.

(1). Fourteen cents is what rate per cent of $2?

Ans. 7.

(2). Two dollars is what rate per cent of $0.14?

Ans. 1428$\frac{4}{7}$.

(3). Twenty-five cents is what rate per cent of 10 cents?

(4). Ten cents is what rate per cent of 25 cents?

(5). What rate per cent of $5 is 37$\frac{1}{2}$ cents?

(6). What rate per cent of $150 is $23.10?

(7). What rate per cent of $8775.50 is $25?

(8). What rate per cent of 1 cent is 16$\frac{2}{3}$ cents?

SECTION II.

Commission.

A person who transacts business for another is called an agent, sometimes a factor or broker.

A commission merchant is an agent who buys or sells goods or property for another, receiving therefor a fee or allowance, called a *commission.*

It often happens that goods are sent from one place to an agent in another. In such case the one who sends goods is called the *consignor;* the one who receives the goods, or to whom they are sent, is called the *consignee,* and the goods sent are called a *consignment.*

The commission is reckoned at some rate per cent of the money used in the transaction for which the commission is allowed. It is important to understand

clearly in each case on what value the commission is to be computed. Thus, an agent sells a house for $8500 at a commission of ½%, and the commission is computed on the whole amount. In this case the whole commission is $42.50. Again, an agent receives $5000 with which to purchase goods, after deducting a commission of 1%. In this case the commission is allowed on account of purchasing, and must be computed, not on $5000, but on what was actually used in purchasing; that is, on what remained after deducting the commission.

Again, an agent is employed to collect taxes enough to leave $10,000, after paying his own commission of 2%. In this case the commission is computed not only on the $10,000, but on $10,000 + commission, because the work of collecting the portion of money used for the commission was of the same nature as for any other portion.

Again, a man sends 10,000 bushels of corn to an agent, who sells it on a commission of 1½%. He also pays expenses of freight and drayage, but no commission is computed on these expenses, because the skill for which commission is allowed is supposed to be used in selling corn.

By the phrase "net proceeds" is meant the amount of receipts left after paying all expenses of commission, freight, and drayage, or whatever other expenses there may be.

Problem.—An agent received $5000 with which to purchase goods after deducting his commission of 1%.

What was the amount of his commission, and how much remained for purchasing goods?

For each dollar used in purchasing goods 1 cent is used for commission, making \$1.01 expended for each dollar's worth of goods bought. As many dollars' worth of goods can be bought as the number of times that \$1.01 is contained in \$5000, or as many as $\frac{5000}{1.01} = 4950.50$; nearly \$4950.50 expended in goods.

of which 1% = 49.50 = commission.

\$5000.00 = whole amount.

Problem.—It is required to know what sum must be assessed in order that \$10000 shall remain after paying a commission of 2% for collecting the whole amount. In this case for each dollar collected only \$0.98 remains after paying the commission.

There must, then, be as many dollars collected as the number of times \$0.98 is contained in \$10000.

$\frac{10000}{.98} = 10204.081 +$. 2% of \$10204.081 = \$204.081 nearly. The whole amount then to be collected is \$10204.08.

This problem may also be solved readily by the method of the Rule of Three. Thus, if it requires \$1 to be collected so that \$0.98 shall remain, how much must be collected that \$10000 shall remain?

$$\$0.98 : 10000 : : \$1 : \frac{10000}{.98} = 10204.08 +.$$

Problem. — A commission merchant sells 10000 bushels of corn at $62\frac{1}{2}$ cents a bushel on a commission of $1\frac{3}{4}\%$. He also pays \$187 for freight and \$125 for drayage. Required to find the net proceeds.

$$10000 \times .62\tfrac{1}{2} = 6250$$

$$\$6250 \times .0175 = \$109.375 = \text{commission.}$$
$$\text{Add freight} = 187.000$$
$$\text{Add drayage} = 125.$$
$$\text{Total expenses} = \$421.375$$
$$\$6250 - 421.375 = 5828.625. \quad \textit{Ans.}$$

PROBLEMS.

(1). An agent sells 150 barrels of flour at \$9.25 ; 19 barrels of molasses, of 32 gallons each, at 75 cents per gallon. Required to find the commission at $1\frac{3}{4}\%$ and the net proceeds.

(2). A farmer sends the veal of 6 calves to market, weighing respectively 133 ℔., 145 ℔., 128 ℔., 163 ℔., 150 ℔., 142 ℔.; also 25 cheeses, weighing altogether 743 ℔. The agent pays \$5.85 for freight and \$1.75 for drayage. He sells the veal of 4 calves, whose weights were named first in order, at 10 cents per ℔., the balance at $11\frac{1}{2}$ cents per ℔. He sells the cheese at $12\frac{1}{2}$ cents per ℔. Allowing a commission of $2\frac{1}{2}\%$, what should the farmer receive as the net proceeds?

(3). A farmer hired a man to harvest his wheat, allowing him 10% for his services, and received 468 bushels. How much did the harvester receive?

(4). A commission merchant received \$1640 with

which to buy corn, after deducting a commission of $2\frac{1}{2}\%$. How many bushels of corn could he buy, and what the amount of his commission?

(5). An agent sells a consignment of pork for \$2164. He pays \$16.50 for freight and storage. A commission of $1\frac{7}{8}\%$ being allowed, what are the net proceeds?

(6). A house and lot were sold for \$1570, of which \$1546.45 were the net proceeds. Required the rate per cent of commission.

(7). An agent received \$31.50 for collecting \$2520. Required the rate per cent of the commission.

(8). The net proceeds of a sale are \$34.50, the commission being \$11.50. Required the rate per cent of the commission.

(9). A commission merchant sells 10000 ℔. of pork at $6\frac{1}{4}$ cents per ℔., on a commission of $2\frac{1}{2}\%$. He also receives \$800 to be invested with the net proceeds of the pork in a stock of dry goods, the commission for buying and selling each at $2\frac{1}{2}\%$. What is the whole commission?

(10). An agent sells a consignment of 14000 ℔. of cotton at 15 cents per ℔. He pays \$25.15 for freight, \$8.35 for drayage, and reserving his commission, remits \$2024.77 as the net proceeds of the sale. Required to find the rate per cent of his commission.

(11). An agent collects 85% of a note of \$175 and charges 3% commission. What are the net proceeds?

(12). A commission merchant sells a consignment of 500 tons of hay on a commission of 5%. He remits \$3650 as the net proceeds, after paying \$150 for freight and drayage. What was the price of hay per ton?

SECTION III.

Stock and Brokerage.

Stock is the capital employed in business conducted by an organized company. It is usually reckoned in shares of $100 each. The owners of the shares are called stockholders. The original cost of a share is called its par value, while the market value is that for which it sells, which may be more or less than its par value.

When a share sells in market for less than its par value, the stock is said to be *below par*, or at a discount; when the market price is more than its par value, it is said to be *above par*, or at a premium.

A dealer in stocks, or one who negotiates the loan of money, is called a *broker*, and the commission in either case is called *brokerage*. In the case of stocks, the commission or brokerage, unless otherwise agreed upon, is computed on the par value of the stock, and not on the price actually paid. Thus a broker buys a share (whose par value is $100) for $90, and receives 2% for brokerage; that is, 2% of $100, not 2% of $90, the money really used. Or if he pays $105 for a share he would receive 2% only on $100, not on $105.

Problem.—A man has $1500 to invest in stocks selling at a discount of 6½% (that is, below par), paying ½% brokerage. How many shares (par value being $100) can he purchase?

For one share he pays \$93.50, and brokerage \$0.25, or altogether \$93.75 for each share. $\frac{\$1500}{\$93.75} = 16.$

Ans. 16 shares.

PROBLEMS.

[NOTE.—In the following questions each share will be reckoned at \$100 each, unless otherwise stated.]

(1). A broker sells 75 shares of stock at 18% below par, charging $\frac{3}{8}$% brokerage. What are the net proceeds?

(2). A broker buys 25 shares of stock, paying a premium of 11%, and charges a brokerage of $\frac{1}{4}$%. Required the total cost.

(3). A broker sells 25 shares of stock at a premium of 11%, charging $\frac{1}{4}$% brokerage. What are the net proceeds?

(4). A broker received \$2000, with instructions to invest in mining stock, remitting any unexpended balance less than the cost of one share. Shares were at a premium of $12\frac{1}{2}$% and brokerage $\frac{1}{4}$%. How many shares did he purchase, and what balance was remitted?

(5). Bought 84 shares of stock at $2\frac{1}{2}$% discount, and sold at a premium of 2%, paying in each case $\frac{1}{2}$% brokerage. What was my gain?

(6). How many shares of bank stock, at 4% premium, can be bought for \$8340, allowing $\frac{1}{4}$% brokerage?

(7). A broker receives \$1480 to invest in stock selling at $92\frac{7}{8}$%. He purchased 16 shares. Required to find the rate per cent of brokerage.

(8). An agent is instructed to buy 50 shares, the market price being $15\frac{1}{2}$% discount, brokerage $1\frac{1}{4}$%. How much money is required?

SECTION IV.

Profit and Loss.

The *profit* of a business transaction is the excess of the selling price above the cost of an article; the *loss* is the excess of cost above the selling price.

Profit and Loss are usually reckoned at a rate per cent on the cost of the article. Thus a drover buys sheep at $2.50 a piece and sells at $3.25. He gains 75 cents on each one, but it is required to find the rate per cent of the gain; that is, to find what rate per cent 75 cents is of $2.50.

$$\frac{.75}{2.50} \times 100 = 30. \quad \textit{Ans. } 30\%.$$

Problems of profit and loss, and of the rate per cent in any case, do not involve any new principle of percentage.

The rate per cent of gain or loss may be found by dividing the gain or loss by the cost price, and multiplying the quotient by 100.

PROBLEMS.

(1). A grocer buys butter at 25 cents per pound and sells at 30 cents. What rate per cent does he gain?

$$.30 - .25 = .05, \text{ gain on 1 lb.}, \ \frac{.05}{.25} \times 100 = 20. \quad \textit{Ans. } 20\%.$$

(2). A grocer gains 5 cents on a pound of butter, and thereby makes a profit of 20%. What did the butter cost?

(3). A grocer sells butter at 30 cents per pound, and makes a profit of 20%. What did the butter cost? [To make a profit of 20% he must sell for 120%. If 30 cents is 120%, what is 100%?]

(4). A man bought a horse for $96, and afterward sold him for $120. What was the rate per cent of gain?

(5). A boy bought a knife for 50 cents and afterward sold it for 25 cents. What was the rate per cent of loss?

(6). A merchant bought 30 gallons of vinegar for $12, and sold it out at 45 cents per gallon. At what rate per cent did he gain or lose?

(7). A grocer buys eggs at 25 cents per dozen and sells at 22 cents. What was the rate per cent of loss?

(8). A merchant buys cloth at $3.00 per yard, and wishes to make a profit of 25%. What must be the selling price?

(9). The same merchant (mentioned in the last example) finding the cloth injured, decides to sell at 15% less than the first asking price. What is the price?

(10). A dealer in real estate sells a house for $8400, gaining 12%. What did the house cost?

(11). The same dealer sold another house for $6600 and lost 12%. What did it cost?

SECTION V.

Insurance.

Insurance is a partial indemnity guaranteed by one party for loss or injury of life or property which may be incurred by another, as the loss or injury of property

by fire, or storms at sea, and the loss of life under various circumstances.

The party giving the guarantee is called the *insurer*, or *underwriter*; the party receiving the guarantee is said to be *insured*.

The *policy* is the written contract between the parties. The *premium* is the sum paid for insurance, and is reckoned as some per cent of the value insured.

Insurance on property is distinguished as *Fire Insurance* and *Marine Insurance*. The first provides for losses by fire, the second for losses at sea.

The value of property provided for by insurance is not often more than two-thirds of the full value, as otherwise the owner might be induced to cause the destruction of property in order to secure its value.

When property which is insured is injured less than the amount guaranteed, the insurers pay the full value of the estimated loss.

Insurance on life is a contract on the part of a company to pay a certain sum, on the death of a person, to his heirs, or it may be a contract to pay the person in case he survives a certain number of years, in consideration of an annual premium, to be paid either during life or for a certain number of years. There is so much variance in the details of the computations required in different cases, and the explanations furnished by insurance companies are so ample, it is not deemed necessary to occupy space here on this branch of the subject. Tables on which such computations are based will be found in the Appendix at the close of the volume.

PROBLEMS.

(1). A house, valued at \$4500, is insured for \$3000, the premium being reckoned at $2\frac{1}{4}\%$. What is the cost of insurance? *Ans.* \$67.50.

(2). A house, being worth \$16000, is insured in one company for \$4000 at 2%, and for \$8000 in another company at $2\frac{1}{2}\%$. What rate per cent of premium is paid on the whole amount insured? *Ans.* $2\frac{1}{8}\%$.

(3). If it cost \$93.50 to insure a store for half its value, at $1\frac{3}{8}\%$, how much is the store worth?

Ans. \$13,600.

(4). A company insured a house at 2%, but reinsured one-half the same amount in another company at $2\frac{1}{4}\%$, when it appeared they had \$87.50 remaining of the premium first received. What was the amount due the owner of the house in case of the loss by fire?

Ans. \$10,000.

(5). A library is insured at a premium of $3\frac{1}{4}\%$ for two-thirds its value, paying \$403. What is the whole value of the library? *Ans.* \$18,600.

SECTION VI.

Duties or Customs.

Duties or customs are taxes paid to the agent of the government by a person importing goods from a foreign country.

Duties may be *ad valorem*, that is, a certain per cent of value in the country from which they are imported, or *specific*, which is a specified sum for each unit

of weight or measure, without regard to value. In all cases allowance is made for waste, loss, or damage.

Tare is an allowance for the weight of the box or covering that contains goods.

Leakage is an allowance on liquors imported in casks or barrels.

Breakage is an allowance for the loss by the breaking of bottles.

Gross weight and value are reckoned before any allowance is made; net weight and value are reckoned after all allowances have been deducted.

All duties taken at the United States custom houses are now *ad valorem*.

Goods are weighed by the British ton of 2240 pounds, or 112 pounds to the hundred-weight.

The bill showing the quantity and price of each kind of goods is called the *invoice*.

PROBLEMS.

(1). A merchant imported 65 casks of wine, of 32 gallons each, at $2.25 per gallon. Freight cost $1.25 per cask, duty 30%, 2% being allowed for leakage; drayage cost $10.50. What was the whole cost of the wine? *Ans.* $5675.58.

(2). An importer paid $750 for an invoice of silks, but damages at 15% were allowed at the custom house, the duty being 24%. What was the entire cost of the goods? *Ans.*

(3). Paid $108 duty on a lot of goods which had

been damaged, the allowance for damage being 20%, and the duty being also 20%. What was the invoice price of the goods? *Ans.*

(4). Paid $680.40 duty on 150 hhd. of molasses, each containing 63 gallons, at 30 cents per gallon. What was the rate per cent of duty. *Ans.*

SECTION VII.

Simple Interest.

Interest is an allowance made for the use of money or its equivalent.

The money, for the use of which interest is allowed, is called the *principal.* Interest is called simple when reckoned only on the principal. It is called compound when reckoned on interest added to principal. The interest is usually reckoned as some rate per cent. of the principal, for each year or for part of a year in the same proportion.

The *amount* is the sum of the principal and interest.

The rates of interest are established by law in the different States, and vary from 5% to 10%. Six % is established in most States and 7% in several others.

When no rate is mentioned in a contract, only the legal rate in the State where the contract is made can be exacted. In some States any rate may be agreed upon, higher than the legal rate, but in other States higher rates are prohibited by law. Any rate of interest higher than the legal rate is called *usury.*

General Problem.

To compute simple interest on any principal, for any time and at any rate per cent of interest. Multiply the principal by the rate per unit, and this product by the number of years.

Thus, to find the interest of \$8 for three years at 8%. Since 8% of \$8 is $\$8 \times \frac{8}{100} = \$.64$, and is the interest for one year, then for three years the interest will be three times as much, or $\$.64 \times 3 = \$1.92$. Suppose the time were 2 years 8 months, this would be reckoned as $2\frac{2}{3}$ years. Usually days are reckoned as 30 to a month; for instance, 5 days, $\frac{5}{30}$ month; 12 days, $\frac{12}{30}$ month, and so on.

As sometimes there are 31 days in a month, and again, in the case of February, only 28 or 29, there is frequently an error in this reckoning, but it is small, and is generally neglected. When it is required to be more exact, any portion of time less than a year will be reckoned in the exact number of days, and 365 days to the year. Thus, \$100 being on interest from April 1, 1876, to July 4, 1877, the time is 1 year 94 days, and interest, at 7%, would be computed $\$100 \times \frac{7}{100} \times 1\frac{94}{365} = \$8.80 +$.

In what follows it will be understood, unless otherwise specified, that a month is reckoned as 30 days and the year as 360 days.

Problem.—Find the interest of \$37.50 for 2 yr. 9 mo. 15 da. at 6%. 2 yr. 9 mo. 15 da. $= 2\frac{19}{24}$ yr. $= \frac{67}{24}$ yr. Hence, $\$37.50 \times \frac{67}{24} \times \frac{6}{100} = \$.09375 \times 67 = \$6.28125 =$ \$6.28 nearly.

Another method of finding interest is to find the interest on \$1 for the given rate and time, and multiply this by the principal. Thus the interest of \$1, at 6%, for

2 years	is	\$.12
6 months	"	.03
3 months	"	.015
15 days, or ½ month....	"	.025
or for 2 yr. 9 mo. 15 da.	"	\$.1675

Multiplying this by the number of dollars in the principal, the result is the same as before. It is probable that this method is more often used in practice than the other, but either method is so simple that it only needs moderate care to insure correct results.

Problem.—Find the interest on \$1040 for 5 yr. 11 mo. 29 da., at 7%. 11 mo. 29 da. = 359 da., and 5 yr. 11 mo. 29 da. $= 5\frac{359}{360}$ yr. $= \frac{2159}{360}$ yr. Hence $\$1040 \times \frac{2159}{360} \times \frac{7}{100} = \$\frac{15717520}{36000} = \$436.597 = \$436.60$ nearly.

Again, it is well to notice that the time given lacks but a single day of being 6 years, and hence it would only be necessary to deduct 1 day's interest from 6 years' interest. But 1 day's interest is $\$1040 \times \frac{7}{100} \times \frac{1}{360} = \$\frac{182}{900} = \$.20\frac{2}{9}$, and 6 years' interest is $\$1040 \times \frac{7}{100} \times 6 = \$436.80$, and the difference is \$436.60 nearly, as before obtained.

Oftentimes much labor of computation may be avoided by taking advantage of considerations which only require thoughtful attention to discover.

It may be remarked that many computers habitually

find the interest first for 6%, then add to or deduct, according as the rate is more or less.

Thus for 7%, first find interest at 6% and then add $\frac{1}{6}$; for 8% add $\frac{2}{6}$, or $\frac{1}{3}$; for 5% deduct $\frac{1}{6}$; for 3% divide by 2, and so on.

When a person has become familiar with the method of finding the interest at 6%, no doubt this may seem the easier way to do.

The following considerations make it a simple thing to compute interest at 6%:

The interest	of	\$1 for 2 months,	at 6%,	is	\$.01
"	"	\$1 for 1 month,	"	"	.005
"	"	\$1 for 6 days,	"	"	.001
"	"	\$1 for 1 day,	"	"	.000$\frac{1}{6}$

Or, for any number of days less than 6, the interest will be so many 6ths of a mill.

To illustrate this, consider the following

Problem.—Find the interest of \$15.25, at 6% for 1 yr. 9 mo. 21 da.

Interest of	\$1 for 1 year, at 6%	\$.06
"	\$1 for 8 months, "	.04
"	\$1 for 1 month, "	.005
"	\$1 for 18 days, "	.003
"	\$1 for 3 days, "	.000$\frac{1}{2}$
Interest of	\$1 for 1 yr. 9 mo. 21 da...	\$.108$\frac{1}{2}$

and $\$.108\frac{1}{2} \times 15.25 = \$1.654625 = \$1.65 +$.

By the general method the operation would be as follows :

$$1 \text{ yr. } 9 \text{ mo. } 21 \text{ da.} = 1 \text{ yr. } 9\tfrac{7}{10} \text{ mo.} = \tfrac{217}{120} \text{ yr.}$$

$\$15.25 \times \frac{6}{100} \times \frac{217}{120} = \$\frac{.1525 \times 217}{20} = \$\frac{33.0925}{20} = \$1.65+$, the same as before.

The students should become familiar with both methods and decide each for one's self which is preferable.

Problem.—A man pays \$10 as interest on \$90 for 1 yr. 3 mo. Required to find the rate per cent per annum.

If we knew the interest of \$90 at 1% for the same time, we should obviously only need to divide the given interest by the interest at 1%. This is easily found.

$$\$90 \times \tfrac{1}{100} \times \tfrac{5}{4} = \$\tfrac{450}{400} = \$1.125. \quad \frac{\$10}{\$1.125} = \tfrac{80}{9} = 8\tfrac{8}{9}.$$

Ans. $8\frac{8}{9}$%.

Problem.—A man pays \$10 interest on \$100, at the rate of 6%. Required to know the time.

Interest of \$100 for one year, at 6%, is \$6, and dividing the given interest by the interest for one year will evidently give the number of years. $\frac{\$10}{\$6} = 1\frac{2}{3}$.

Ans. 1 yr. 8 mo.

PROBLEMS.

[The rate of interest is understood to be 6% per annum, unless otherwise specified.]

(1). Find the interest of $1200 for 1 yr. 3 mo., and also the amount.

Ans. Interest is $90 and the amount is $1290.

(2). Find the amount of $75.15 for 2 yr. 9 mo. 15 da.

(3). Interest of $824.36 for 5 yr. 5 da?

(4). Interest of $15.625 for 10 days?

(5). Interest of $.01 for 1 day? for 10 days?

(6). Interest of $95.95 for 19 mo. 19 da. at 5%?

(7). Interest of $1000 for 2 yr. 6 mo. at 4%?

(8). Amount of $1475 for 5 yr. 5 mo. 5 da. at 10%?

(9). Interest of $375.375 for 13 days at $6\frac{1}{2}$%?

(10). Interest of $1250 for 63 days at 12%?

(11). Interest of $95.05 for 119 days at $12\frac{1}{2}$%?

(12). Amount of $25.50 for 17 days at 15%?

(13). Amount of $2550 for 34 days at 15%?

(14). Interest of $150.49 for 10 mo. 10 da. at $4\frac{1}{2}$%?

(15). Interest of $157.85 from March 1, 1869, to July 5, 1872, at 8%?

(16). Amount of $2100 from Dec. 4, 1871, to Feb. 4, 1872, at $7\frac{3}{10}$%, reckoning 365 days to the year?

(17). Amount of 99 cents for 99 days at 6%?

(18). Interest of $137.75 for 1 yr. 11 mo. 13 da. at $8\frac{1}{2}$%?

(19). Amount of $185.85 for 3 yr. 5 mo. 15 da. at $3\frac{1}{2}$%?

(20). Interest of $28145.30 for 5 yr. 11 mo. 25 da. at 11%?

SECTION VIII.

Notes, with Indorsements.

A promissory note is a written statement of an obligation to pay money on demand, or at some future time. The form of a note often varies in the details, but the following will answer for illustration:

$2500. NEW YORK CITY, June 1, 1877.

Two years after date I promise to pay to John Jones, or order, twenty-five hundred dollars, with interest at 6%, for value received. RICHARD DOE.

An *indorsement* is a written acknowledgment on the back of a note, stating the time and amount of a partial payment made on account of the note.

According to decisions of the United States Supreme Court, when a partial payment exceeds the interest then due, the excess is applied to diminish the principal of the note. If the payment falls short of the interest due the principal of the note is not disturbed, no interest is reckoned on the payment, but the former principal of the note continues to draw interest until subsequent payments are made, such that the sum shall exceed the interest due on the principal of the note.

These decisions are embodied in the following UNITED STATES RULE:

I. *Find the amount of the given principal to the time of the first payment, and if this payment exceed the interest then due, subtract the payment from the amount obtained, and treat the remainder as a new principal.*

II. *But if the interest be greater than the payment, compute the interest on the same principal to a time when the sum of the payments shall equal or exceed the interest due, and subtract the sum of the payments from the amount of the principal; the remainder will form a new principal, with which proceed as before.*

The above rule is used in nearly all the States. There are slight variations established in Vermont, New Hampshire, and Connecticut, but they present no special difficulties.

Problem.—A note was given as follows:

$\$150\frac{75}{100}$. ST. LOUIS, January 4, 1870.

Eighteen months after date I promise to pay Andrew Jackson, or order, One hundred fifty and $\frac{75}{100}$ *dollars, with interest at* 10%, *value received.*

J. B. WHITMAN.

It was indorsed as follows: Nov. 4, 1870, $\$10\frac{00}{100}$. Nov. 1, 1871, $\$25\frac{00}{100}$. March 10, 1872, \$75. How much was due June 1, 1873?

Operation.

First principal, Jan. 4, 1870	$150.75
Interest to Nov. 4, 1870, more than the payment $10..	12.562+
Interest from Jan. 4, 1870, to Nov. 1, 1871	27.512
Amount	$178.262
Deduct sum of payments, $10+$25 = ...	35.00
	$143.262
Interest from Nov. 1, 1871, to March 10, 1872	5.133
Amount	$148.395
Deduct third payment	75.00
	$73.395
Interest from March 10, 1872, to June 1, 1873	8.991
Ans. Amount	$82.386

PROBLEMS.

$1000. BOSTON, MASS., Oct. 1, 1870.

(1). Value received. I promise to pay John Bond, or order, one thousand dollars on demand, with interest, at 6%. RICHARD BLAKE.

Indorsed as follows: June 4, 1871, $50. May 10, 1872, $35. Oct. 4, 1872, $80. April 1, 1873, $50. Oct. 4, 1873, $100. What remained due Dec. 1, 1874?

Ans. $922.08+.

1850\frac{75}{100}$. LAWRENCE, KANSAS, May 1, 1864.

(2). Six months after date, I promise to pay A. Newman, or order, eighteen hundred fifty and $\frac{75}{100}$ dollars, with interest at 8%. Value received. JOHN DOE.

Indorsed as follows: Sept. 20, 1865, received $250.50. Oct. 25, 1867, received $300.90. July 11, 1869, re-

ceived $85.70. Sept. 20, 1870, received $212.10. Dec. 5, 1871, received $400. How much was due May 1, 1872?

$3475. ST. LOUIS, November 4, 1870.

(3). On or before May 4, 1871, I promise to pay George Evans, or order, thirty-four hundred seventy-five dollars, with interest after said date at 6%. Value received. AUGUSTUS TOMPKINS.

Indorsed as follows: May 4, 1871, received $475. Nov. 1, 1871, received $200. Dec. 15, 1872, received $100. Dec. 18, 1873, received $200. June 4, 1874, received $575. May 4, 1875, received $1200. How much was due May 4, 1876?

SECTION IX.

Compound Interest.

Interest computed on principal and accrued interest is called *compound.*

When interest becomes due and is not then paid, it is in some cases added to the principal, and thus becomes a source of compound interest.

It is usually regarded as illegal, but is allowed in some cases.

Usually, when it is allowed, the interest is reckoned as annually due, and the interest is then added or compounded at the end of each year.

Thus, what is the compound interest of $100, for 4 years, at 6%?

Interest of \$100 for 1 year	\$6.00
Amount for 1st year and principal for 2d year.........................	\$106.00
\$106 × 1.06 = amount for 2d year.....	112.36
\$112.36 × 1.06 = amount for 3d year..	119.10+
\$119.10 × 1.06 = amount for 4th year.	126.247+
	100.00
	26.247

Deduct the first principal, \$100, and there remains \$26.25, nearly, as the compound interest.

The method requires the amount at the close of one year to become the principal for the next year.

If the interest were payable semi-annually, or quarterly, as is sometimes the case, then the amount due at the end of six months, or at the end of the quarter, would become the principal for the next period.

The difference between the first principal and the amount finally due is regarded as constituting the compound interest.

PROBLEMS.

(1). What is the interest of \$500, for 5 years, at 5%, compounded annually? *Ans.* \$138.14+.

(2). What is the interest of \$500, for 5 years, at 5%, compounded semi-annually? *Ans.* \$140.04+.

(3). What will \$250 amount to in 5 years, at 7%, compound interest? *Ans.* \$350.638+.

(4). What will \$250 amount to in 5 years, at 7%, compounded semi-annually? *Ans.* \$352.65, nearly.

(5). Find the compound interest of \$188 for 3 yr. 8 mo. 15 da., at 6% annually. [In this case the amount is found for 8 mo. 15 da., as in any case, and the first principal deducted.] *Ans.* \$45.42.

Tables to facilitate the computation of compound interest, showing the amount of \$1 at various rates of interest and for various periods of time, are easily prepared, but seldom used.

SECTION X.

Discount.

Discount is an allowance made for the payment of a debt before it becomes due.

The *present worth* of a debt is such a sum as, placed at interest, will amount to the given debt when it becomes due.

Problem.—A man owes \$100, to be paid in one year, but wishes to discharge the debt at once. What is the present worth of this debt, reckoning interest at 6%?

Since the present worth would amount to \$100 if placed at interest, it must be such a sum that, multiplied by 1.06, it would produce \$100. Hence $\frac{\$100}{1.06} =$ \$94.34 nearly, is the present worth, and the discount is $\$100 - \$94.34 = \$5.66$.

It is clear, in any case, that the present worth may be found by dividing the debt by the amount of \$1 *for the given rate and time. The discount is the difference between the given debt and the present worth.*

PROBLEMS.

(1). What is the present worth and the discount at 7% of $2875, due in 1 yr. 4 mo. 20 da.?

(2). A man owes $5000, due in 9 months, and $5000 due in 15 months. Money being worth 10%, what is the present worth of the two debts?

(3). A man bought a house on May 1st, agreeing to pay $1200 in 4 months and $2000 in 9 months. Were he to make payment at once, 10% discount of the whole sum would be allowed. Supposing he could borrow money at 6% per annum, how much would he thereby save?

(4). A man bought 100 bales of cotton, each reckoned at 500 pounds, at 9 cents a pound, on 6 months' credit, and at once sold the cotton for $4950, and paid the debt, at a discount of 10% per annum. How much did he realize?

(5). A man can borrow money at 8% per annum, and wishes to buy flour, which is offered at $6 per barrel cash, or $6.50 on 9 months' credit. Which should the man do?

SECTION XI.

Banking.

A bank is an institution, authorized by law, for the purpose of receiving and loaning money, and in some cases to issue notes (called bank-notes) to circulate as money.

A promissory note has been defined (Section VIII.)

as a written statement of an obligation to pay a certain sum of money, on demand, or at some future time.

The one who signs the obligation is called the maker or drawer of the note, and the person to whose order the note is made payable is called the payee.

The *face* of a note is the sum specified in the note to be paid.

Days of grace are three days usually allowed for the payment of a note after the expiration of the time mentioned in the note.

A note *matures* when it is *legally due*; that is, when days of grace are allowed, at their expiration; or, in States where no grace is allowed, at the time specified in the note.

Bank discount is an allowance made, according to the usage of banks, for the payment of a note before it becomes due. It is reckoned as the interest on the face of the note, computed from the date of the note to the date of its maturity.

The *proceeds* of a note is the sum received for it when discounted, and is found by deducting the bank discount from the face of the note.

A note is said to be *discounted* when taken at a bank in exchange for its computed value.

As a matter of custom, banks do not often discount notes running more than 6 months, and the actual number of days in any calendar month is usually counted in making up the reckoning.

If the date of maturity falls on Sunday, or on a legal holiday, a note must be paid on the day preceding.

To indicate when a note is due, the date specified in

the note is written at the left, the date of maturity on the right of a vertical line; thus, May 4 | 7.

Problem.—Required to compute the date of maturity and the proceeds of the following note, discounted at 6%.

$1000. ALBANY, March 10, 1872.

Ninety days after date, we promise to pay to the order of John Smith, one thousand dollars, at the Exchange Bank. Value received. BAKER & Co.

Reckoning the days in each month, it is easily seen that 90 days extend to June 9th, and 93 days extend to June 12th.

The interest of \$1000, for 93 days, at 6%, is $\$1000 \times \frac{6}{100} \times \frac{93}{360} = \$\frac{31}{2} = \$15.50$, the discount, and $\$1000 - \$15.50 = \$984.50$, the proceeds.

Ans. Due June 9 | 12; proceeds, \$984.50.

Problem.—Required to compute the date of maturity and the proceeds of the following note, discounted at 6%.

$1000. ALBANY, March 10, 1872.

Three months after date, we promise to pay to the order of John Smith, one thousand dollars, at the Exchange Bank. Value received. BAKER & Co.

In this case three months extend to June 10, although this includes 91 days, and the note matures June 13th.

Hence $\$1000 \times \frac{6}{100} \times \frac{94}{360} = \$\frac{47}{3} = \$15.66\frac{2}{3}$, the discount, and $\$1000 - \$15.66\frac{2}{3} = \$984.33\frac{1}{3}$, the proceeds.

Ans. Due June 10 | 13; proceeds, $\$984.33\frac{1}{3}$.

Problem.—Required to find the face of a note discounted at 6%, for 90 days, whose proceeds are \$1000.

Here it is understood that the note was due in 93 days.

First let us find the discount on \$1, for the same rate, and for the same length of time.

$\$1 \times \frac{6}{100} \times \frac{93}{360} = \$.0155$. the discount, and the proceeds \$.9845. Here it seems that $98\frac{45}{100}$ cents, as the proceeds, require \$1 on the face of a note. It is clear that \$1000, as proceeds, would require as many dollars on the face as $\frac{1000}{.9845} = \$1015.74+$. *Ans.* \$1015.74+.

A proportion would also express the relation in a similar manner:

$$\$.9845 : \$1000 :: \$1 : \frac{\$1000}{.9845} = \$1015.74+.$$

That is, the proceeds of a dollar are to the required proceeds as the face \$1 to the required face.

In any case, to find the face of a note whose proceeds shall equal a given sum, divide the given sum by the proceeds of \$1 for the same time and rate.

Problem.—A man has a 30 days' note for \$100 discounted at the bank, receiving therefor \$99.45. It is required to find the rate per cent per annum of interest paid on the sum actually received. In this case the borrower received \$99.45, and at the end of 33 days pays the amount \$100, or 55 cents interest for the use of \$99.45 for 33 days. The interest of \$99.45 for 33 days, at 1% per annum, is $\$99.45 \times \frac{1}{100} \times \frac{33}{360} = \$\frac{1093.95}{12000}$ $= \$\frac{1.09395}{12} = \$\frac{.36465}{4} = \$.0911625$.

The true rate per cent will then be the quotient of the actual interest, divided by the interest at 1%; that is, $\frac{.55}{.0911625} = 6.03 +$, or a little more than $6\frac{3}{100}$%.

Problem.—A banker is offered a note of $100 which matures in 33 days. He wishes to discount the note at such rate per cent that he will realize an interest of 10% per annum on his investment.

As the rate will be the same, without reference to the face of the note, let us suppose the face to be such that the proceeds shall be $1. In 33 days, at 10%, this will amount to 1.009\frac{1}{6}$. Hence this must be the face of the note, and the discount is $.009$\frac{1}{6}$, and $.009$\frac{1}{6}$ is the interest of 1.009\frac{1}{6}$ at some rate per cent to be found.

But the interest of 1.009\frac{1}{6}$, at the rate of 1% per annum for 33 days, is $$\frac{.13321}{144}$, and dividing the given interest by this, we shall evidently have the rate required; that is,

$$.009\tfrac{1}{6} \times \frac{144}{.13321} = \frac{.11}{12} \times \frac{144}{.13321} = \frac{13.2}{1.3321} = 9\tfrac{12111}{13321}.$$

Ans. $9\frac{12111}{13321}$%.

In any case, then, to find the rate per cent of discount that shall secure a given rate of interest upon the proceeds of a note:

Find the interest and amount of $1, *at the required rate, for the given time.*

Find the interest on this amount for the same time at 1%, *and by this divide the interest of* $1 *before found.*

PROBLEMS.

(1). Find the date of maturity and the proceeds of the following note, discounted on the day given, at 2% a month, or 24% per annum :

$1250. BROOKLYN, January 31, 1872.

Value received. One month after date, I promise to pay William Brown, or order, twelve hundred fifty dollars, at the Globe Bank. PETER TRUSTY.

Due Feb. 28 | Mch. 3. Proceeds, 1224.16\frac{2}{3}$.

(2). Find the proceeds of the following note, discounted the date at which the note was given, at 6%:

$853.45. CHICAGO, June 15, 1871.

Sixty days after date, value received, I promise to pay Ezekiel Jones, or order, eight hundred fifty-three $\frac{45}{100}$ dollars, at the Union Bank. CHARLES SCRIBNER.

(3). What are the proceeds of a 90 days' note for $1305, discounting at 8%?

(4). What is the face of a note at 90 days, of which the proceeds, discounted at 7%, are $1000?

(5). For what amount must one draw a note at 40 days, discounting at 1% a month, that the proceeds shall be $175?

(6). For what amount must I draw at 60 days to obtain $500 proceeds, when discounted at 2% a month?

(7). What rate of interest does the borrower pay on the sum used, when a note for 30 days is discounted at 10%, and what at 12%?

(8). What rate of interest does the borrower pay, discounting a note of 90 days, at 8% ?

(9). What is the rate of discount when the banker realizes 12% on a note of 30 days ?

(10). What must be the rates of discount to realize on notes of 60 days 8%, 9%, and 10% respectively ?

SECTION XII.

Exchange.

Exchange is a method of obtaining money in one place, by means of written orders, in consideration of money deposited in another place.

A *bill of exchange*, or a *draft*, is a written demand by one person, upon a second person, to pay a specified sum to a specified person, at a specified time.

A draft requiring payment "at sight" is called a *sight* draft, or bill. In other cases, the time specified is a certain interval, usually a number of days after date.

The one who signs an order or draft is called the *drawer* or *maker*. The one to whom the order is addressed is called the *drawee*. The one to whom the money is directed to be paid is called the *payee*.

The payee may be the drawer, but is usually a third person, and the bill is said to be drawn "in favor of" the payee.

A bill may be made payable to "*A B*, or bearer," or it may be made payable "to bearer," and in either case, whoever presents the bill to the drawee is entitled to receive the specified sum.

Usually, however, a bill is made payable "to A B, or order," in which case A B (the payee), writing his name upon the back of the bill, and presenting it to the drawee, is entitled to receive the specified sum.

The payee having written only his name upon the back of the bill, the drawee is obligated to pay it, by whomsoever presented.

The payee may transfer his interest in a bill by writing an order on the back of the bill directing payment to be made to some other person.

The signature of the payee upon the back of the bill, whether with or without an order, is called an *indorsement*.

Each one who indorses a draft becomes separately responsible for the amount of the bill, in case the drawee fails to make payment.

A bill is *accepted* when the drawee writes the word "accepted," with his signature, across the face of the bill, and he thereby promises to pay the bill when it becomes due.

The course of exchange is the price paid in one place for drafts or bills of exchange on another place. This price depends upon the prevailing current of traffic between the two places. If the imports to New York from Liverpool exceed in value the exports from New York to Liverpool, then bills of exchange on Liverpool will command a higher price in New York, because specie must be shipped from New York to pay for the balance of trade.

In *domestic or inland exchanges* only one kind of money is considered, and the course of exchange is ex-

pressed by a certain rate per cent of premium or discount. In case of a draft on time, the usual bank discount is allowed for the interval of time.

In the case of foreign exchange, two (or more) kinds of money are considered, and the course of exchange is expressed by stating a certain value of the coin of one country, in some denomination of the other country, as the rate at which exchanges are to be effected. Thus, if exchange on London is quoted at $4.87\frac{1}{2}$, it means that £1 is reckoned as equivalent to $\$4.87\frac{1}{2}$. If the exchange on Paris is quoted at 5.18, it means that $5\frac{18}{100}$ francs are reckoned as equivalent to \$1.

Three copies of a bill of exchange, called a set of exchange, are usually made in the case of a foreign remittance, and sent at different times or by different means, as security against miscarriage. Any one having been paid, the others become void.

The *intrinsic par of exchange* expresses the comparative value of the coins of two countries, as determined by the actual weight and purity of the metal. The *commercial par of exchange* expresses the comparative value of the coins of two countries as reckoned in commercial transactions.

The intrinsic par of exchange is constant so long as the coin is unchanged, but the commercial par is liable to frequent changes.

Exchange is said to be direct when only one bill, or one set of exchange is used in one remittance.

Exchange is "arbitrated" when it is effected by using an intermediate bill or set of exchange.

Thus one may remit from Boston to Paris direct by

sending a draft payable in Paris; or one may remit by arbitration, sending a bill payable in London to an agent there, and direct him to send a bill payable in Paris to the payee there.

Sometimes the exchange is effected through two or more intermediate points.

In any case, if *A* is to receive money from *B*, *A* may draw upon *B* and sell the bill, or *B* may remit a bill made in favor of *A*.

English money was formerly called "sterling" money, and bills of exchange on cities of England, Ireland, or Scotland are called sterling bills, or sterling exchange. (The name sterling is supposed to be derived from Easterling, a name once given in England to German traders, whose money was supposed to be of the purest quality.)

By an enactment of 1873 the weight and fineness of United States coins were established as follows:

Coins.	*Weight.*		*Fineness.*
Eagle, gold	258	grains.	.900.
Dollar, gold...........	25.8	"	.900.
Trade dollar, silver....	420	"	.900.
Half dollar, silver	12½	grammes.	.900.
Five cents, nickel.....	17.16	grains.	.75 copper. .25 nickel.
Three cents, nickel....	30	"	.75 copper. .25 nickel.
One cent, bronze......	48	"	.95 copper. .05 tin and zinc.

According to the above the gold dollar weighs 25.8 grains, but of this only 23.22 grains are gold, the balance, 2.58 grains, being silver. The value of this silver

is not included in the worth of the dollar, the gold itself being worth the dollar. It would follow from the foregoing that an ounce (480 grains) of pure gold would be worth $20.672. The half dollar does not contain half the amount of silver contained in the trade dollar, and the relative value of silver fluctuates according to the relations of demand and supply.

The commercial values of foreign coins may be found in a table of the Appendix.

Exchanges with Europe are usually made through some of the following-named cities: London, Paris, Antwerp, Berlin, Bremen, Hamburg, Frankfort, and Amsterdam.

In bills of exchange on Paris and Antwerp the money specified is named in francs; for Berlin, Bremen, Hamburg, and Frankfort, in marks; and on Amsterdam, in guilders.

The computation of direct inland exchange is according to simple principles of percentage, and requires no special rules.

In the computation of arbitrated exchange it is recommended that the operations of reduction be first indicated, so that cancelling may be advantageously used.

The following examples will illustrate:

Problem.—What must be paid in Boston for a draft on Baltimore, at 30 days, for $2500, exchange being at ½% premium, and bank discount reckoned at 6%?

Let us first find what must be paid for $1.

$$
\begin{array}{rrl}
\$1+.005 = & \$1.005 & \text{course of exchange.} \\
 & .0055 & \text{bank discount for 33 days.} \\
\hline
 & \$0.9995 & \\
\end{array}
$$

$\$0.9995 \times 2500 = \$2498.75$ is the cost required.

Problem.—Required to find the cost of a bill of exchange on London, the course of exchange being 4.87, as follows:

£500. NEW YORK, August 20, 1877.

At sight of this first of exchange (second and third of same date unpaid), pay to the order of A. Livingston, New York, five hundred pounds, value received, and charge the same to account of

BAUM & BERGER.

To JOHN ABERNATHY & Co.,
London, England.

As £1 would be $4.87, of course £500 would cost $\$4.87 \times 500 = \$2435$.

Problem.—What will it cost to send a bill of 5000 marks to Hamburg, first sending a bill of the proper amount to an agent in Paris, the exchange in New York on Paris being 5.15 francs per dollar, and the exchange in Paris on Hamburg being 1.24 francs per mark?

If one mark is worth 1.24 francs, then 5000 marks = 1.24×5000 fr. = 6200 francs. If 1 franc is worth $\frac{1}{5.15}$

of a dollar, then 6200 francs $= \$\frac{6200}{5.15} = \$1203.88+$. Or the whole operation may be indicated at once.

$$\frac{1.24 \times 5000}{5.15} = 1203.88+.$$

Problem.—What must be paid in St. Louis for a sight draft of \$100 on Boston, exchange being at $\frac{1}{2}\%$ premium?

Let us first find the cost for \$1.00, then multiply by the number of dollars required.

The cost of \$1.00 is \$1.005. The cost of \$100 is $\$1.005 \times 100 = \$100.50$. *Ans.*

Problem.—What must be paid in Albany for a draft on Minneapolis, drawn 90 days, for \$1200, at a premium of $1\frac{3}{8}\%$, reckoning bank discount at 7%?

$$\begin{array}{ll} \text{Course of exchange of \$1} & = \$1.01375 \\ \underline{\text{Discount of \$1, 93 days,}} & = \underline{.01808\frac{1}{3}} \\ \text{Cost of \$1} & = \$0.99566\frac{2}{3} \end{array}$$

$$\$0.99566\tfrac{2}{3} \times 1200 = \$1194.800$$

PROBLEMS.

(1). What is the cost of a draft for \$275.00 for 30 days, reckoning bank discount at 12%, and the course of exchange being $1\frac{1}{2}\%$ discount?

(2). What is the cost of a draft for \$965.00 for 60 days, bank discount at 8%, and the course of exchange $\frac{1}{2}\%$ premium?

(3). A sight draft cost \$506.63$\frac{5}{8}$, exchange being at a premium of 1$\frac{1}{8}$%. What was the face of the draft?

(4). A banker in New York remits \$4000 to London as follows: first to Paris, at 5.32 francs per dollar; thence to Hamburg, 1.8 francs per mark; thence to London, at 14 marks per £1 sterling. What is the amount of available funds in London, and how much would be gained by this means over direct exchange at \$4.87 per £1 sterling?

Ans. £844 8s. 10$\frac{3}{8}$d. in bank; a gain of £23 1s. 9$\frac{5}{12}$d.

(5). A merchant in New Orleans wishes to remit \$5000 to New York. The direct exchange is 1$\frac{1}{2}$% premium, but the exchange on St. Louis is $\frac{1}{2}$% premium, and between St. Louis and New York is 1% discount. What will it cost to remit by the way of St. Louis, and how much will be saved?

(6). What must be paid in Savannah for a draft on Boston, at 60 days, for \$850, exchange at $\frac{3}{4}$% discount, and bank discount reckoned at 8%?

(7). A banker in New York, wishing to remit to his correspondent in Paris, directs his agent in London to draw on him for \$1000 and remit to Paris. Exchange in London was at the rate of £1 per \$4.86, and 25 francs per £1. Allowing the agent 1$\frac{1}{4}$% commission, both for drawing and remitting, to be deducted, how much did the correspondent receive in Paris?

(8). A resident of Vienna wishes to remit 1000 florins to his son in New York; buys a draft on Berlin at 1.8 marks per florin and sends to the son. He sells this at the rate of 4 marks per \$1. What does he realize?

CHAPTER VI.

Equation of Payments.

When a debtor has to make several payments, due to the same creditor at different dates, he may wish to pay the whole amount at one time, such that no loss shall result to either party.

The method of finding the date at which the amount of several debts may be paid at once, without loss to debtor or creditor, is called the "Equation of Payments," or sometimes the "Average of Payments."

Different methods are used for this purpose, and there is some difference of opinion as to the absolute accuracy of these methods, according to what is regarded as the basis of equity. But generally these errors are so small that they may be neglected.

A simple method in common use will be understood from the following illustration:

A owes B $500.00, to be paid June 1st; $800.00, to be paid October 1st, and $700.00 to be paid on the 1st of December. Required to find the average date of payment.

Reckoning from the earliest date, June 1st, for convenience, we may consider that A is not entitled to the use of the $500.00 any longer, but is entitled to use

\$800.00 for four months (to Oct. 1st), and entitled to the use of \$700.00 for six months (that is, to Dec. 1st).

But the use of \$800.00 for 4 months is equivalent to the use of $\$800 \times 4 = \$3200$ for 1 month, and the use of \$700 for 6 months is equivalent to the use of $\$700 \times 6 = \$4200$ for 1 month. In other words,

$$
\begin{array}{rrcr}
 & \$500 \times 0 & = & \$00.00 \\
 & 800 \times 4 & = & 3200.00 \\
 & 700 \times 6 & = & 4200.00 \\
\hline
\text{Whole amount} = & \$2000, & &)\$7400.00\,(3\tfrac{7}{10}
\end{array}
$$

That is, A is entitled to the equivalent of the use of \$7400 for 1 month, and the question is, how long may he have the use of \$2000 in order that such use shall be equivalent to this?

This is evidently expressed by the quotient of $7400 \div 2000$; that is $3\frac{7}{10}$ months, or 3 months and 21 days, because the use of \$2000 for $3\frac{7}{10}$ months is equivalent to the use of $\$2000 \times 3\frac{7}{10} = \$7400$ for 1 month.

The above computation is based upon the supposition that the use of \$500 for 1 month, after it is due, is balanced by the payment of \$500, 1 month before it is due. This would be true, if bank discount were a true discount.

But, as before stated, the error may be neglected.

The above reckoning was made in months, but it will often be more convenient to reckon in days. If a fractional part of a day appears in the result, it will be neglected if less than $\frac{1}{2}$, or it will be counted as 1, if $\frac{1}{2}$ or more than $\frac{1}{2}$.

The method deduced from the foregoing may be described as follows:

Reckoning from the earliest date at which any debt becomes due, multiply each debt by the number of months (or days) which elapse before it becomes due. Divide the sum of the products by the sum of the debts, and the quotient expresses the interval of time to elapse before the average date of payments.

Problem.—On the 4th March I purchase a bill of goods for $200, to be paid in 30 days, and another bill of $150, to be paid in 60 days. On the 20th March I purchase a bill of $300, to be paid in 30 days. Required to find the date of average payment.

Ans. April 17th.

Problem.—Bought, May 4th, on 60 days' credit, a bill of $250; May 20th, on 90 days' credit, a bill of $300; and on June 1st, 30 days' credit, a bill of $500. Required to find the average date of payment.

Ans. July 15th.

The balance of an account, containing both debits and credits, may be settled on the same principle as the foregoing.

Suppose A bought goods of B on the 1st of May, on 30 days' credit; on the 15th of May, at cash rates, a bill of $400; and on the 25th of May, on 60 days' credit, a bill of $600. Suppose on the 20th of May he pays $200 cash, June 1st $250, and on the 15th of June

\$250. Required to find the time at which he may equitably pay the balance.

In this case \$400 are due May 15th, which is the earliest date; \$300 are due May 31st, and \$600 due July 24th. Whence we have

$$\begin{array}{rr} \$400 \times 0 = & 00.00 \\ 300 \times 16 = & 4800.00 \\ 600 \times 70 = & 42000.00 \\ \hline \$1300 \quad & \$46800.00 \end{array}$$

That is, he was entitled to the equivalent of the use of \$46800 for 1 day.

But reckoning from the same date to the dates of payments, and multiplying the sums by the respective numbers of days, we have

$$\begin{array}{rr} \$200 \times 5 = & \$1000.00 \\ 250 \times 17 = & 4250.00 \\ 250 \times 31 = & 7750.00 \\ \hline \$700 \quad & \$13000.00 \end{array}$$

And it seems he has paid \$700, which, however, he retained long enough to be equivalent to the use of \$13000 for 1 day:

$$\begin{array}{rr} \$1300 & \$46800.00 \\ 700 & 13000.00 \\ \hline \$600 & \$33800.00 \end{array}$$

There remains, then, \$600 to pay, which may be retained long enough to be equivalent to the use of \$33800 for 1 day; that is $\frac{33800}{600} = 56\frac{1}{3}$, or omitting $\frac{1}{3}$, 56 days, to be reckoned from May 15th; that is, the balance of the account should be paid July 10th.

PROBLEMS.

(1). Required to find the proper date for paying the balance due on the following account:

GEORGE FORD, *in account with* RIDENOUR & BAKER.

Dr. *Cr.*

1878.					1878.				
April	10	To Mdse. (Cash price)	$50	00	April	15	By cash..........	$100	00
"	15	" " 30 days....	250	00	May	15	"	200	00
"	25	" " 30 days...	300	00					

Ans. May 29, 1878.

(2). Required to find the equitable date for paying the balance of the following account:

RIDENOUR & BAKER *in account with* GEORGE LEWIS.

Dr. *Cr.*

1877.					1877.				
May	24	To Mdse., 30 days...	$85	00	May	31	By cash..........	$100	00
June	2	" " Cash rates.	100	00	June	2	"	100	00
"	15	" " 30 days....	200	00	July	1	"	100	00

[NOTE.—For convenience, reckon the dates from May 31st.]

Ans. Sept. 5, 1877.

CHAPTER VII.

Alligation.

It is sometimes required to find the value of a mixture of several ingredients of different values; or, again, it may be required to make a mixture of a definite value from ingredients of known values. In either case, the process is known as Alligation, which means, in a literal sense, a "linking together," and the process, in this case, is probably so called from the manner in which problems of this class are frequently wrought, as may be seen in some of the examples that follow.

CASE I.

To find the value of a mixture when values and quantities of the ingredients are known.

Problem.—A grocer mixes 3 pounds of sugar, worth 10 cents per pound, with 2 pounds of sugar worth 15 cents per pound. It is required to find the value per pound of the mixture.

It is evident that

3 pounds,	@ 10 cents,	cost	30 cents.
2 "	@ 15 "	"	30 "
and 5 pounds of mixture		cost	60 cents.

Or 1 pound of the mixture cost 12 cents.

In any such case, then, *divide the entire cost of the mixture by the entire number of measures used*, to find the cost of one measure of the mixture.

PROBLEMS.

(1). A merchant paid $1 per bushel for 20 bushels of wheat, $1.20 per bushel for 15 bushels, and 90 cents per bushel for 40 bushels. What is the average cost per bushel of the whole amount? *Ans.* $98\frac{2}{3}$ cents.

(2). A grocer mixed 10 pounds of dried leaves, which cost 5 cents a pound, with 10 pounds of tea at 60 cents and 8 pounds of tea at 75 cents per pound. Required to find the value of 1 pound of the mixture.

Ans. $44\frac{9}{14}$ cents.

(3). If 1 lb. 9 oz. of gold, 23 carats fine, be compounded with 2 lb. 4 oz. of 21 carats, 1 lb. 10 oz. 10 pwt. at 20 carats, and 1 lb. 1 oz. of alloy, what will be the fineness of the composition? *Ans.* 18 carats.

CASE II.

Given the values of the ingredients and the required value of the mixture—to find the quantity of each ingredient.

Problem.—A grocer has sugars at 10 cents and at 15 cents per pound, and wishes to make a mixture worth 12 cents a pound. It is required to find how much of each kind may be used.

For each pound of 10 cent sugar used the mixture

would lack 2 cents; for each 2 pounds used the mixture would lack 4 cents, and for each 3 pounds used the mixture would lack 6 cents of the required value. Again, for each pound of 15 cent sugar used, the mixture would exceed the required value by 3 cents, and for each 2 pounds used the value would be 6 cents in excess. In other words, in mixing 3 pounds at 10 cents with 2 pounds at 15 cents, the excess of the required value will just balance the deficiency. Thus

3 pounds	@ 10 cents	cost	30 cents.
2 "	@ 15 "	"	30 "
5 pounds	of mixture	cost	60 cents,
or 1 pound	"	"	12 "

And it will be noticed that the difference between each given price and the required price indicates the number of parts to be used of the other kind.

This is illustrated by the following diagram:

$$\text{Required value} = 12 \begin{cases} 10 \\ 15 \end{cases} \begin{matrix} \text{3 at 10 cents.} \\ \text{2 at 15 cents.} \end{matrix}$$

It is evident that a mixture of more or less than 5 pounds may be made, provided the kinds be used in the same proportion. Thus, if 2 or 3 times as much of one kind be used, then 2 or 3 times as much of the other kind must be used. In this last problem, for instance, if 4 pounds instead of 2 pounds, at 15 cents, be used, then 2 times 3 pounds, at 10 cents, must be used. Or if $\frac{1}{2}$ pound ($\frac{1}{4}$ of 2), at 15 cents, be used, then $\frac{1}{4}$ of $3 = \frac{3}{4}$ pound, at 10 cents, must be used; and it is easy to

see that the price of the mixture will still be 12 cents per pound.

Again, it may be required to make a specified amount of the entire mixture. In the same problem, for instance, it may be required to make a mixture of 40 pounds. Dividing 40 by 5 (the sum of the proportional parts found), the quotient 8 shows that 8 times each of the proportional parts will be required; that is, 8 times $2 = 16$ pounds at 15 cents, and 8 times $3 = 24$ pounds at 10 cents. Or if 24 pounds of the whole mixture were required, then $\frac{24}{5} \times 2 = 9\frac{3}{5}$ pounds at 15 cents, and $\frac{24}{5} \times 3 = 14\frac{2}{5}$ pounds, at 10 cents, would be used.

Further, there may be three or four or more kinds to be mixed, but the average price will be found by comparing 2 at a time, one less and the other greater than the required price.

Thus, if the grocer had sugar at 14 cents to mix with the other two named, the comparison would be made as indicated in the following:

$$12 \left\{ \begin{array}{l} 10 \\ 14 \\ 15 \end{array} \right. \quad \begin{array}{ll} 3+2 & = 5 \text{ at } 10 \text{ cents}, = 50. \\ 2 & = 2 \text{ at } 14 \text{ cents}, = 28. \\ 2 & = 2 \text{ at } 15 \text{ cents}, = 30. \\ \hline & \text{Or } 9 \text{ at } 12 \text{ cents} = 108. \end{array}$$

Here it appears that comparing 10 cent with 14 cent sugar, 2 pounds of each may be used to give the average of 12 cents, and at the same time using the parts previously found, 2 pounds at 15 cents, with 3 pounds

at 10 cents, the result is as seen above: 5 pounds, 2 pounds, and 2 pounds respectively.

Suppose, further, the grocer had sugar at 11 cents, making 4 kinds altogether, still requiring a mixture worth 12 cents per pound.

The comparisons indicated in the following will be evident:

12	10	3 at 10	cents	cost	30 cents.
	11	2 at 11	"	"	22 "
	14	1 at 14	"	"	14 "
	15	2 at 15	"	"	30 "
		8 of	mixture	cost	96 cents
		Or 1	"	"	12 "

Or as follows:

12	10	2 at 10	cents	cost	20 cents.
	11	3 at 11	"	"	33 "
	14	2 at 14	"	"	28 "
	15	1 at 15	"	"	15 "
		8 of	mixture	cost	96 cents.
		and 1	"	"	12 "

Whenever, then, it is required to make a mixture of a certain value, *compare the value of each ingredient of less than the required value with one greater, and use as many parts of each as indicated by the differ-*

ence between the other value and the required value, taking care, if any value is compared with more than one other, to use the sum of the differences.

Remember, also, if the amount of the entire mixture is limited, or if the amount of one ingredient is limited, in either case it is necessary to *use the different ingredients in the same relative proportion as first found.*

PROBLEMS.

(1). How much each of teas, worth respectively 30, 36, 40, 50 cents per pound, may be used to obtain a mixture worth 42 cents per pound?

How much to make a mixture of 100 pounds?

Ans. 24 pounds at 50 cents, and 8 of each of the others. For 100 pounds in all, use 50 pounds at 50 cents, and 16$\frac{2}{3}$ pounds of each of the others.

(2). A farmer mixes corn at 40 cents with oats at 30 cents, barley at 50 cents, and rye at 80 cents, to make a mixture worth 48 cents per bushel. In what proportion may he arrange the parts, having only 20 bushels of rye? *Ans.* 80, 5, 45, and 20 bushels.

(3). It is required to find how much gold 20 carats fine must be compounded with 10 pounds of gold 1·' carats fine, to obtain a compound 18 carats fine.

Ans. 20 pounds.

(4). Alcohol, 96% strong, contains 4% of water and 96% of pure spirits. It is required to find how much water must be mixed with 1 gallon of alcohol, 96% strong, so that the mixture shall contain 80% of pure spirit. *Ans.* $\frac{1}{5}$ gallon.

(5). A trader bought 100 bushels of corn at 48 cents, 200 bushels at 50 cents. Required to find how much he must buy at 40 cents that the average price of the whole will be 45 cents per bushel. *Ans.* 260 bushels.

(6). Required to find how much pure gold shall be mixed with 1 oz. gold 12 carats fine and 2 oz. of 16 carats to obtain a compound of 18 carats fine.

Ans. $1\frac{2}{3}$ oz.

CHAPTER VIII.

Arithmetical and Geometrical Series or Progressions.

A SERIES is a set of numbers, increasing or decreasing in a regular manner, so that any number of the series may be deduced from one or more of those that precede, by some rule.

An increasing series is sometimes called ascending, and a decreasing series descending.

There are many kinds of series, but only two will be considered here.

I. AN ARITHMETICAL SERIES

Is one in which the numbers increase or decrease by a constant (or common) difference.

Thus, 5 ·· 8 ·· 11 ·· 14, etc., is an increasing arithmetical series, in which the common difference is 3. 102 ·· 98 ·· 94 ·· 90 ·· 86, etc., is a decreasing arithmetical series, in which the common difference is 4.

The numbers which form a series are called terms, and there are five elements of a series that may be subject to computation. These are: the first term, the common difference, the number of terms, the last term, and the sum of the terms. It will be seen that any

three of these being known, the other two may be found. The first and last terms are often called the extremes.

Suppose it were required to find the last term in the series of which the first term is 5, the common difference is 3, and the number of terms is 10.

It is obvious that

1st term	$= 5$	
2d "	$= 8$	$= 5+3$
3d "	$= 11$	$= 5+2\times 3$
4th "	$= 14$	$= 5+3\times 3$
5th "	$= 17$	$= 5+4\times 3$
etc.	etc.	

And it is evident that any term is equal to the first term + the common difference multiplied by the number of terms that precede. Hence the 10th term $= 5 + 9\times 3 = 32$.

In general, to find the last term in an increasing arithmetical series, *multiply the common difference by one less than the number of terms, and to the product add the first term.*

In a decreasing series the product of the common difference multiplied by one less than the number of terms, *must be subtracted from the first term.*

Suppose, again, it were required to find the sum of ten terms of the same series.

$$5 \cdot\cdot 8 \cdot\cdot 11 \cdot\cdot 14 \cdot\cdot 17 \cdot\cdot 20 \cdot\cdot 23 \cdot\cdot 26 \cdot\cdot 29 \cdot\cdot 32.$$

It is evident from inspection that 1st term + last term $= 5+32 = 37$. Term following 1st + term pre-

ceding last $= 8+29 = 37$. 2d term after 1st + 2d term before last $= 11+26 = 37$, etc.

It is easy to see that the sum of the first and last term is equal to the sum of any two others, one following the first, the other preceding the last in the same order, because one is more than the first, and the other is less than the last by the same number of common differences. It is further evident that the number of such sums is half the number of terms. In case of an odd number of terms the middle term forms half a sum.

In the present case there are $\frac{10}{2} = 5$ such sums, each equal to 37, and the sum of the ten terms is therefore $(5+32)\times\frac{10}{2} = 37\times 5 = 185$.

As the same considerations apply to all similar cases, it follows that *the sum of the terms in any arithmetical series is equal to the sum of the first and last terms, multiplied by half the number of terms.*

Since the last term is found by adding the first term to the product of the common difference multiplied into one less than the number of terms, it follows that the difference between the first and last terms must always be the product of those two factors, the common difference and the number of terms less one. Since either factor may be found by dividing the product by the other factor, it follows, to find the common difference, we may *divide the difference of the extremes by one less than the number of terms*, or to find the number of terms, *divide the difference between the extremes by the common difference, and increase the quotient by* 1.

PROBLEMS.

(1). In the series of odd numbers 1 ·· 3 ·· 5 ·· 7 ·· 9 ·· etc., required to find the 47th term. *Ans.* 93.

(2). In the series $\frac{1}{2}$ ·· 2 ·· $3\frac{1}{2}$ ·· 5 ·· $6\frac{1}{2}$ ·· etc., required to find the 20th term, and the sum of the 20 terms.

Ans. 29, 295.

(3). In the series 101 ·· 98 ·· 95 ·· 92 ·· etc., required to find the 30th term, and the sum of the 30 terms.

Ans. 14, 1725.

(4). A farmer sold 30 sheep as follows: the first one for 25 cents, the second one for 50 cents, the third for 75 cents, and so on. Required to find what he obtained for the lot. *Ans.* $116.25.

(5). Required to find the sum of the first 100 whole numbers in the natural order, 1, 2, 3, etc.; also the sum of the first 100 odd numbers, 1, 3, 5, 7, etc.; also the sum of the first 100 even numbers, 2, 4, 6, 8, etc.

Ans. 5050, 10000, 10100.

(6). In a series, of which the first term is 15, the number of terms is 15, the last term is 57, it is required to find the common difference. *Ans.* 3.

(7). In a decreasing series, of which the first term is 50, the common difference 5, the last term 10, it is required to find the number of terms. *Ans.* 9.

(8). A laborer laid aside $150 of his wages, at the close of each year, for 10 years, at 8% per annum, simple interest. Required to find the amount of his savings at the end of 10 years. *Ans.* $2040.

II. A GEOMETRICAL SERIES,

Is one in which the numbers increase or decrease in a constant ratio.

Thus 1, 2, 4, 8, 16, etc., is an increasing geometrical series in which the ratio is 2, and 1, $\frac{1}{2}$, $\frac{1}{4}$, $\frac{1}{8}$, etc., is a decreasing series in which the ratio is $\frac{1}{2}$.

There are five elements in a geometrical series which may be considered in computation: *the first term, the ratio, the number of terms, the last term, and the sum of the terms*, any three of which being known the other two may be found.

The number of terms may be infinite; that is, continue without end; and if the series is increasing the terms must finally become infinite. If the series be decreasing the terms must finally become zero.

Thus, in the series 1, $\frac{1}{2}$, $\frac{1}{8}$, $\frac{1}{16}$, etc., if extended to infinity, the terms must evidently become smaller and smaller, and finally reach the limit zero.

Let it be required to find the last term in a geometrical series of which the first term, the ratio, and the number of terms are known.

For instance, to find the 10th term in the series 8, 16, 32, 64, etc.

From inspection it appears that

1st term $= 8$.
2d " $= 8 \times 2$.
3d " $= 8 \times 2^2$.
4th " $= 8 \times 2^3$.

and it is evident that *any term is equal to the product*

of the first term multiplied by the ratio raised to a power less by 1 *than the number of terms.*

In the present case the 10th term $= 8 \times 2^9 = 8 \times 512 = 4096$.

Or again, to find the 8th term in the series $1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}$, etc., in which the ratio $= \frac{1}{2}$, it is easy to find 8th term $= 1 \times (\frac{1}{2})^7 = 1 \times \frac{1}{128} = \frac{1}{128}$.

If the last term were known, and it were required to find the first term, it would only be necessary to invert the order of the terms, and consider the last term of the given series as the first term of a new series, in which the ratio would be the reciprocal of the given ratio.

Thus, if the number of terms is 10, the last term 4096, and the ratio 2, to find the first term; we may invert the order, regarding 4096 as the first term of a new series, of which the ratio is $\frac{1}{2}$.

Then the 10th term $= 4096 \times (\frac{1}{2})^9 = 4096 \times \frac{1}{512} = 8$, the first term of the given series which was sought.

Suppose, again, it were required to find the ratio when the extremes (the first and last terms) and the number of terms are known.

It is evident from what precedes that the quotient of the last term, divided by the first, is equal to a power of the ratio of an order less by 1 than the number of terms. Hence, to find the ratio, *divide the last term by the first, and find the root of the quotient, of an order* 1 *less than the number of terms.*

Again, if the extremes and the ratio are known, to find the number of terms, it is evidently sufficient to

divide the last term by the first, and then find by trial what power of the ratio is equal to this quotient.

Suppose, finally, it were required to find the sum of the terms of a geometrical series.

For instance, to find the sum of 6 terms in the series 8, 24, 72, etc.—1944.

It is evident that if each term be multiplied by the ratio 3, the sum of all the products would be 3 times the sum of the given series.

Performing the multiplication, the result may be written as below:

Sum of series $= 8+24+72+\text{etc.}\cdots+1944.$
3 times sum of series $= 24+72+\text{etc.}\cdots+1944+5832.$

It appears from inspection that the terms of each are the same, excepting the first term of the first series and the last term of the other.

Subtracting one series from the other, there would remain 2 times sum of series $= 5832 - 8 = 5824$, or the sum of the series $= \frac{5824}{2} = 2912.$

As the same kind of reasoning may be applied in other like cases, it follows that the sum of any number of terms of an increasing geometrical series may be found by multiplying the last term by the ratio, from this product subtracting the first term, and dividing the difference by the ratio less 1.

If the series is decreasing, subtract the product of the last term multiplied into the ratio, from the first term, and divide this by 1 less the ratio.

That is, in either case, *divide the difference between the first term and the product of the last term multi-*

plied into the ratio, by the difference between the ratio and 1.

Thus to find the sum of 10 terms of the series, 3, $\frac{3}{10}$, $\frac{3}{100}$, etc. $\cdot\cdot$ $\frac{3}{1000000000}$, in which the ratio is $\frac{1}{10}$. The foregoing precept gives the sum of

10 terms $= (3 - \frac{3}{10000000000}) \div (1 - \frac{1}{10}) = \frac{29999999997}{10000000000} \div \frac{9}{10} = \frac{3333333333}{1000000000} = 3.333333333.$

The same method is applicable when the series extends to infinity. In that case the last term is 0.

Suppose we have the series $1 + \frac{1}{2} + \frac{1}{4} +$ etc., extending to infinity, the ratio being $\frac{1}{2}$.

Then the sum of the series $= (1 - 0) \div (1 - \frac{1}{2}) = 1 \div \frac{1}{2} = 2.$

PROBLEMS.

(1). Find the 10th term in the series 64, 32, 16, etc. *Ans.* $\frac{1}{16}$.

(2). Find the 8th term in the series, of which the first term is 3 and the ratio 3. Find the sum of the same terms. *Ans.* 6561, 9840.

(3). The extremes are $\frac{1}{2}$ and $40\frac{1}{2}$, and the number of terms 5; to find the ratio. *Ans.* 4.

(4). The first term is 5, the last term is $244\frac{9}{64}$, the number of terms is 7; to find the ratio. *Ans.* $2\frac{1}{2}$.

(5). Given the first term 4, the ratio $2\frac{1}{2}$, the last term $976\frac{9}{16}$; to find the number of terms. *Ans.* 7.

(6). Required to find the sum of 12 terms of the series 4, 6, 9, etc. *Ans.* $1029\frac{421}{512}$.

(7). Find the value of $.55\dot{5}$, regarded as an infinite series, the first term being $\frac{5}{10}$ and the ratio $\frac{1}{10}$. *Ans.* $\frac{5}{9}$.

(8). Find in a similar way the value of $.0\dot{3}$, $.\dot{2}\dot{7}$, $.\dot{1}0\dot{8}$.

Ans. $\frac{1}{30}$, $\frac{3}{11}$, $\frac{12}{111}$.

(9). An annuity (a sum of money due annually, or at regular intervals) of $500 has not been paid for 6 years. Allowing compound interest at 8% per annum, what is the entire sum due for 6 years?

Ans. $3667.95.

(10). A man travels a journey in 10 days, traveling 10 miles the first day, 15 miles the second day, 20 miles the third day, and so on, increasing the distance each day by 5 miles. How far did he travel in the 10 days?

Ans. 325 miles.

CHAPTER IX.

Mensuration.

SECTION I.

Definitions.

Mensuration is the process of measuring the size or extent of bodies, and relates to measures of lines, surfaces, and volumes, and includes all necessary computations therefor.

It forms a part of Geometry, which extends over a wider field of research.

Lines, surfaces, and volumes have been previously defined.

Direction is the tendency of a point moving along a line.

A *point* is regarded in mensuration as place or position, without magnitude or extent.

This position is often indicated by a visible dot, in the same manner that the place of a line which has no width is represented by a visible mark made by pen, pencil, or crayon.

A straight line is one that lies wholly in one direction, and, properly speaking, there are two straight lines between any two points, A and B, for instance, as a line from

A ———————— B

A to B, and a line from B to A, one opposite to the other in direction.

A plane is a flat surface extending without limit.

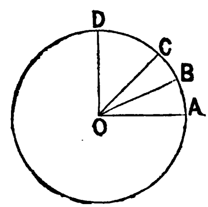
An angle is the difference of direction of two lines, and when the lines meet the point of meeting is called the vertex of the angle.

A right angle is half the angle of opposite directions. When one line meets another so that the divergence on one side equals that on the other, each of the two equal angles is a right angle.

An angle less than a right is called *acute;* one greater than a right is called *obtuse.*

Acute and obtuse angles are both called oblique. An angle is proportioned to the arc of a circumference, included between its sides, in case the center of the circumference is at the vertex of the angle.

Thus, in the figure, if the arc AC is twice the arc AB, then the angle AOC is twice the angle AOB.



As the arc of a circumference is measured in degrees, so the angle, because it is proportional to the arc, is also reckoned in degrees. The right angle being half of opposite directions, or one-fourth the angle of the entire circumference, is counted as 90°, or one-fourth of 360°, the circumference.

A vertical line is one whose direction is that of a plumb-line, freely suspended and at rest.

A horizontal line is one at right angles to a vertical one. A horizontal line is represented on the page of a

book by a line extending from left to right, while a vertical line is represented by a line from the top towards the bottom of the page. This is often called a vertical line, as the picture of a horse may be called a horse; but a line is truly vertical only when it has the direction of a plumb-line.

Two straight lines are said to be *parallel* when they have the same direction. They cannot meet, however far they are produced, for in that case they could not have the same direction at the point of meeting. They must continue at the same distance apart, since they can neither converge nor diverge, and it may be shown that they must both be in one plane.

A plane figure is a portion of a plane surface bounded by straight or curved lines.

A polygon is a plane figure bounded by straight lines. Polygons may have any number of sides greater than two, but in this book those of three sides and those of four sides will only be considered.

A polygon of three sides is called a *triangle*, a polygon of four sides is called a *quadrilateral.*

A triangle having a right angle is called a right-angled triangle, or simply a right triangle. Thus, in Fig. 1, ABC is a right-angled triangle, the vertex of the right angle being at A. The longest side of a right-angled triangle is opposite the right angle, and is called the *hypothenuse*, as in BC in the figure.

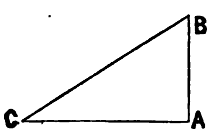


FIG. 1.

The base of a triangle is the side on which it is assumed to rest, but any side may be taken for the base.

In the figure, CA would be regarded as the base and AB the perpendicular.

A triangle of three equal sides is called *equilateral*, as ABC in Fig. 2.

A triangle of only two equal sides is called *isosceles*. as in Fig. 3, where AB = BC.

A triangle of which no two sides are equal is called *scalene*, as ABC in Fig. 4.

The altitude of a triangle is the length of a line

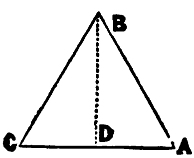


FIG. 2.

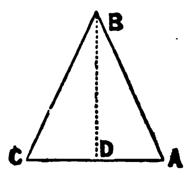


FIG. 3.

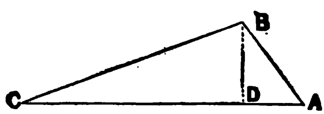


FIG. 4.

drawn from the vertex of the triangle (that is, the vertex of the angle opposite the base) to the base.

The dotted line, BD, in each of the last three figures, represents the altitude.

A quadrilateral may have its opposite sides parallel, when it is called a *parallelogram*; it may have two sides only parallel, when it is called a *trapezoid*; or it may have no two sides parallel, when it is called a *trapezium*.

Figures 5 and 6 represent parallelograms; Fig. 7 a trapezoid; Fig. 8 a trapezium.

A right-angled parallelogram (Fig. 5) is called a

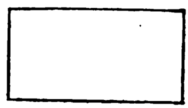


FIG. 5.

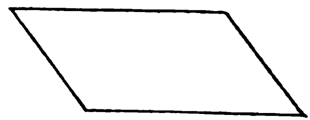


FIG. 6.

rectangle; an oblique-angled parallelogram is called a *rhombus* (see Fig. 6).

A rectangle of equal sides is called a *square* (see

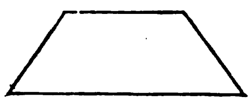


FIG. 7.

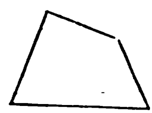


FIG. 8.

Fig. 9), and a rhombus of equal sides is called a *rhomboid* (see Fig. 10). It is also shown in Geometry that the opposite sides of a parallelogram are *equal* as well as parallel.

A circle is a plane figure bounded by a curve, called

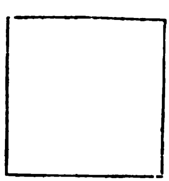


FIG. 9.

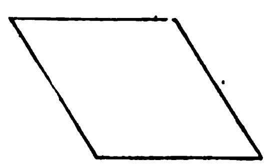


FIG. 10.

the circumference, every point of which is equally distant from a point within, called the center.

The *diameter* of a circle is a straight line passing through the center, terminating at each end in the circumference.

The *radius* of a circle is any line extending from the center to the circumference, and is equal to half the diameter.

SECTION II.

Areas of Quadrilaterals and Triangles.

The area of a rectangle (as stated in Chapter III., Part II.) is expressed by the product of the units in length multiplied into the units in breadth.

It is easily shown that the area of any oblique parallelogram is equal to that of any rectangle of the same base and equal altitude.

Thus, suppose ABCD, Fig. 11, to be a parallelogram, and ABIK to be a rectangle with the same base and equal altitude. It is easy to see that KI = DC, since each is equal to AB, and that KD = IC, and that the area AKD = area BIC since one triangle could be placed upon the other so as to coincide with it. If from the whole figure, ABCK, the triangle AKD be taken away, there remains the parallelogram ABCD; while if the equal triangle BIC be taken away from the same figure, there remains the rectangle ABIK. If equals be taken from equals, the remainders must be equals, and the area of the parallelogram is equal to that of a rectangle of the same base and equal altitude.

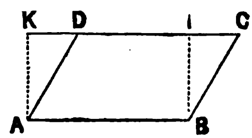


FIG. 11.

It may also be shown that the area of a triangle is half that of a parallelogram whose base and altitude are respectively the same.

For let ABC, Fig. 12, be any triangle, and form the triangle CBD so that CD = AB, and BD = AC, the side BC being common, we shall then have the parallelogram ABDC, composed of two equal triangles of the same base and altitude as the parallelogram, and

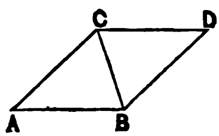


FIG. 12.

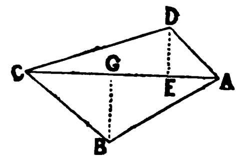


FIG. 13.

of course the area of one triangle will be half that of the parallelogram.

Hence, if the number of units in the base of a triangle be multiplied by half the units in the altitude, the product will express the area of the triangle.

The same thing is often stated more briefly: "The area of a triangle is equal to half the product of the base multiplied into the altitude."

To find the area of any trapezium, as ABCD (Fig. 13), it is necessary to measure a diagonal as AC, and the perpendiculars dropped on this diagonal from the other corners of the figure, as DE and BG in Fig. 14. Then the areas of the triangles may be found separately, and added together. This will be the product of the diagonal into half the sum of the perpendiculars.

To find the area of a trapezoid. Let ABCD be any trapezoid, AB the longer base, CD the shorter base. Draw CP parallel to the side AD, forming the parallelogram APCD, and PB will be the difference in the length of the two bases of the trapezoid. Draw CI to the middle of PB, and draw IG equal and parallel to CP, and extend

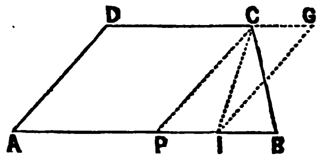


FIG. 14.

DC to G, making CG equal and parallel to PI. Then the parallelogram PIGC equals the triangle CPB in area, and the parallelogram AIGD equals the trapezoid in area. But AI, the base of this parallelogram, equals half the sum of the two bases. Hence the area of the trapezoid is equal to the product of half the sum of the two bases into the altitude.

The following proposition of Geometry has frequent applications in problems of mensuration:

The area of the square constructed on the hypothenuse of a right-angled triangle is equal to the sum of the areas of the squares constructed on the other two sides.

In Fig. 15, let ABC be any right triangle, right angled at A, and BC the hypothenuse. On the line AC take AE equal to AB, and construct the square ABDE; this will be one of the squares. Then prolong AC to H so that CH shall equal AE, and EH will equal AC. Prolong ED to G so that EG will equal EH, and

form the square EGKH, each side of which will be equal to AC, and we have the equal of the second square described in the proposition.

Now the right-angled triangle, BDL, has the side BD, the right angle at D, and the side DL, equal respectively to the side BA, the right angle at A, and the side AC of the triangle BAC, and one can be placed upon the other so that they will coincide, and their

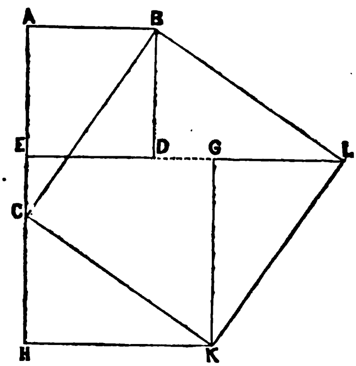


FIG. 15.

areas must be equal. In a similar way LGK can be shown to be equal to CHK.

If, then, from the figure HABDGK, two triangles, BAC and CHK, be taken away, and then two equal triangles, BDL and LGK, be added, the resulting area will be the same; that is, the area CBLK will equal the area HABDGK, the sum of the two squares. But each side of CBLK is equal to BC, the given hypothenuse, because each is the hypothenuse of an equal right-angled triangle, and CBLK is a square constructed

on the hypothenuse whose area is equal to the sum of the areas constructed on the other two sides of the given triangle.

The area of a square is expressed by the square of the number of units in one side. Hence the square of the number of units in the hypothenuse of a right-angled triangle is equal to the sum of the squares of the numbers of units in the two sides.

For example, suppose the two sides of a right triangle to be respectively 3 and 4 feet in length, and it is required to find the length of the hypothenuse. We have $3^2 = 9$, and $4^2 = 16$, and the sum $9 + 16 = 25$. Hence the square of the number of feet in the hypothenuse is 25, and therefore the hypothenuse itself is 5 feet.

PROBLEMS.

(1). Required to find the number of square feet in a triangular board 18 feet long, 15 inches wide at one end and tapering to a point at the other.

$$\tfrac{1}{2} \times 18 \times \tfrac{15}{12} = \tfrac{45}{4} = 11\tfrac{1}{4}.$$

Ans. $11\frac{1}{4}$ square feet.

(2). How many acres in a triangular piece of land of which the base is 100 rods, and the length measured in a line perpendicular to the base, 320 rods?

Ans. 200 acres.

(3). The base of a triangle is 9 feet 6 inches, its altitude is 18 feet 4 inches. Required the area.

Ans. $87\frac{1}{12}$ square feet.

(4). A triangle has an area of $87\frac{1}{12}$ square feet; the base is 9 feet 6 inches. What is the altitude?

(5). The area of a triangular field is 1 A. 2 R. 10 sq. rd.; the base of the field is $12\frac{1}{2}$ rd. What is the length? *Ans.* 40 rods.

(6). The altitude of a trapezoid is 15 feet, the bases measure respectively 23 and 37 feet. Required the area. *Ans.* 450 square feet.

(7). A field in the form of a trapezoid measures, in a perpendicular between the parallel sides, 86 rods, the length of one base is 157 rods, the other 87 rods. Required the area. *Ans.* 65 A. 2 R. 12 rd.

(8). The two sides of a right triangle are 6 feet and 8 feet respectively. What is the length of the hypothenuse?

(9). The length of the hypothenuse is 10 feet, the length of one side is 8 feet. What is the length of the other side?

(10). A ladder 39 feet in length is placed obliquely against a vertical wall so that the top reaches to a height of 36 feet from the foot of the wall. Required to find the distance of the foot of the ladder from the foot of the wall, the line being horizontal.

(11). Required to find the altitude and area of an isosceles triangle, whose equal sides are each 13 feet, and whose base is 12 feet in length.

(12). The diagonal of a quadrilateral measures 80 feet, and the perpendiculars, dropped from the other two vertices upon the diagonal, measure respectively 19 and 25 feet. Required the area.

Ans. 1760 square feet.

It is demonstrated in Geometry that the area of a triangle, of which the three sides are known, may be found by the following method:

Divide the sum of the three sides by 2. From this quotient subtract each side separately, and multiply together the quotient and the three remainders, and take the square root of the product.

Thus the sides of a triangle are respectively 13 feet, 20 feet, and 21 feet; it is required to find the area.

$13+20+21=54$, $\frac{54}{2}=27$, $27-20=7$, $27-21=6$, $27-13=14$.

$27\times 7\times 6\times 14=81\times 14\times 14=9^2\times 14^2$. $\sqrt{9^2\times 14^2}=9\times 14=126$.

Ans. 126 square feet.

Problem.—Required the area of a triangle whose sides are respectively 24, 32, and 40 feet.

Ans. 384 square feet.

SECTION III.

The Circle.

To find the ratio of the diameter of a circle to the circumference is a problem that occupied the attention of mathematicians for a long time, and the student, when further advanced, will no doubt find the account of the matter very interesting. Here we must be content with a statement of the result. It has been proved that the exact value of this ratio cannot be expressed in numbers, but the approach to accuracy may be made as near as any difference that can be named.

If the diameter of a circle be 1 foot in length, the circumference will measure 3.1416 feet nearly; or if the diameter be 2 feet the circumference will measure 2×3.1416 feet nearly, and so on. The value 3.1416 is the one commonly used, and is true within 1 of the fourth decimal order, and answers all practical purposes, though a value has been ascertained true to the 500th decimal order.

To find the circumference of any circle, multiply the diameter by 3.1416; to find the diameter of a circle when the circumference is known, divide the circumference by 3.1416.

To find the area of a circle we may suppose radii drawn from the center, so near together that the arc between any two consecutive ones would seem like a straight line; then the portion of the circle between the same radii (which is called a sector) would become almost a triangle. The area of one such triangle would be the product of its base multiplied by half its altitude (that is, by half the radius) and the sum of all such triangles, which would be the whole circle, would be the sum of all the bases; that is, the whole circumference multiplied by half the radius. This, indeed, expresses the area of the circle, *half the product of the circumference multiplied by the radius.*

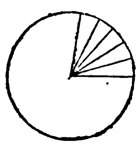


FIG. 16.

Or since the radius is half the diameter, we may say one-fourth the product of the circumference multiplied by the diameter; or again, as the circumference equals the diameter $\times$ 3.1416, the area of a circle equals one-

fourth the square of the diameter $\times 3.1416 =$ diameter $\times .7854$.

Problem.—What is the area of a circle whose diameter is 5 feet? $5^2 \times .7845 = 19.635$. *Ans.* 19.635 sq. ft.

Problem.—The area of a circle is 19.635 square feet. What is the diameter?

Since the square of the diameter is multiplied by .7854 to obtain the area, then dividing the area by .7854 must give the square of the diameter. Hence $\frac{19.635}{.7854} =$ $25 =$ square of diameter, and $\sqrt{25} = 5 =$ diameter.

Ans. 5 feet.

Problem.—The area of a circle is 490.875 sq. rods. What is the radius?

$\frac{490.875}{.7854} = 625$. $\sqrt{625} = 25 =$ diameter, and $\frac{25}{2} =$ $12\frac{1}{2} =$ radius.

Problem.—A circular field contains 10 acres. Required to find the diameter. *Ans.* 45.136+ rods.

SECTION IV.

Similar Figures.

Two figures, having the same form, and whose sides, taken in the same order, are proportional, are said to be *similar*.

Thus 2 rectangles, one 2 feet wide and 6 feet long, the other 8 feet wide and 24 feet long, are similar,

since the length of each is 3 times the width. The area of the first is $2 \times 6 = 12$ square feet, that of the second $8 \times 24 = 192$ square feet; that is, 16 times the first, because in multiplying together the length and breadth of the second, 4 times the length of the first is multiplied by 4 times the breadth; that is, 16 times the product of the first length and breadth.

In general, the surfaces of similar figures are to each as the squares of the corresponding sides. For another illustration: suppose a triangular field has 10 acres inclosed; another field of the same shape, but each side 3 times as long, will contain $3 \times 3 = 9$ times 10 acres.

All squares are similar, all circles are similar, all equilateral triangles are similar.

PROBLEMS.

(1). A triangular field, whose base measures 20 rods, contains 5 acres. How many acres does a similarly shaped field contain whose base measures 60 rods?

(2). A triangular field, whose base measures 20 rods, contains 5 acres; another similar field 45 acres. What is the length of its base? What the base of a similar field which contains 80 acres?

SECTION V.

Solids.

A solid is a body which occupies some volume. When it is bounded by plane surfaces these are called faces, and their intersections are called edges.

A *prism* is a solid, two of the faces of which, called

bases, are equal polygons, in parallel planes, the other faces being parallelograms.

Prisms are named from the figures of their bases, as a triangular prism, a quadrangular prism, and so on.

The altitude of a prism is the perpendicular distance between the bases.

A *parallelopipedon* is a prism of six faces, the opposite ones, two by two, being equal and in parallel planes.

A *cube* is a parallelopipedon whose faces are all equal squares.

A *cylinder* is a body such as would be described by a rectangle revolving about one side. The ends are circles equal and in parallel planes, and any sections made by parallel planes are equal circles.

A *pyramid* is a body whose base is a polygon, and whose other faces are triangles meeting in a common vertex.

A *cone* is a body such as would be described by a right triangle revolving about one side. One end would be a circle, the other would be a point called the vertex, and any section made by a plane parallel to the base would be a circle smaller than the base, according to the distance of the section from the vertex.

The *frustum* of a pyramid, or of a cone, is the portion included between the base and any section parallel to the lower base, the section in each case being called the upper base.

The *altitude* of a pyramid or cone is the perpendicular distance from the vertex to the plane of the base.

A *right pyramid* is one whose vertex is in a line per-

pendicular to the base and passing through the center, the base being a regular polygon, that is, one of equal sides and equal angles.

The *slant height* of a right pyramid is the perpendicular distance from its vertex to one of the sides of the base.

The *slant height* of a cone is the distance from the vertex to any point of the circumference of its base.

A *sphere* is a body bounded by a curved surface, all the points of which are equally distant from a point within, called a center.

The *diameter* of a sphere is a straight line passing through the center of a sphere, each end terminating in the surface

The *radius* of a sphere is a straight line drawn from the center to the surface of the sphere.

Problem.—Required to find the lateral surface of a triangular prism whose altitude is 10 feet and whose base is an equilateral triangle, each side of which is 1 foot.

The prism would have three lateral faces, each face being a rectangle, of the same altitude, and the sum of the areas of these faces would equal the sum of the bases multiplied by the common altitude. That is, $3 \times 10 = 30$. *Ans.* 30 feet.

It is obvious, in any case, the lateral surface of any prism is equal to the sum of the sides of the base; that is, the perimeter of the base multiplied by the altitude.

Problem.—Required to find the convex surface of a right cylinder of which the altitude is 10 feet and the

circumference of the base 3 feet. It is clear that any thin fabric, as paper, cut into the form of a rectangle 3 feet wide and 10 feet long, could be wrapped or folded around the given cylinder in such manner as to exactly cover the convex surface. Hence the convex surface of this cylinder would be found by multiplying the altitude by the circumference of the base, $3 \times 10 = 30$. It is evident that the same method is applicable to any cylinder.

Problem.—Required to find the convex surface of a cylinder of which the altitude is 4 feet, and the radius of the base of which is 4 feet. *Ans.* 100.5312 sq. feet.

Suppose it were required to find the area of the lateral surface of a right pyramid. The area of any face is equal to half the base multiplied by the slant height. The area of all the faces will evidently be equal to half the sum of all the bases of the faces multiplied by the slant height; that is, equal to half the perimeter of the base of the pyramid multiplied by the slant height.

Problem.—Required to find the lateral surface of a right pyramid of which the slant height is 10 feet, and each side of the triangular base of which is 6 feet.

Ans. 90 feet.

Again, if the number of the sides of the base of a pyramid were increased in number indefinitely, the perimeter of the base would more and more approach the form of the circumference of a circle, and the pyramid would approach the form of a cone, and the

lateral surface would tend to become the convex surface, until finally there would be no sensible difference, and the convex surface of the cone would be found by multiplying half the circumference of the base by the slant height.

Problem.—Required to find the convex surface of a cone, the radius of the base of which is 10 feet, and the slant height of which is 8 feet. *Ans.* 251.328 sq. feet.

To find the lateral surface of one face of the frustum of a pyramid which is a trapezoid, we multiply half the sum of the two bases by the slant height. The surface of all the faces would evidently be equal to half the sum of the perimeters of the two bases multiplied by the slant height.

The same reasoning would apply here as in seeking the surface of the cone, and the surface of the frustum of a cone would be equal to half the sum of the circumferences of the two bases multiplied by the slant height.

Problem.—Required to find the lateral surface of the frustum of a pyramid of four faces, each side of the lower base of the frustum measuring 4 feet 6 inches, and each side of the upper base 3 feet 6 inches, the slant height 9 feet. *Ans.* 144 square feet.

Problem.—Required to find the convex surface of the frustum of a cone, the radius of the lower base of which is 25 inches, the radius of the upper base 15 inches, and the slant height 2½ feet.

Ans. 26 sq. ft. 26 sq. in. nearly.

It is proved in Geometry that the surface of a sphere is equal to the diameter multiplied by the circumference of a great circle of the sphere.

A great circle of the sphere is a section made by a plane passing through the center. It is a circle of which the radius is the same as the radius of the sphere.

Problem.—Required to find the surface of a sphere of which the radius is 5 feet. Diameter $= 10$ feet, circumference $= 31.416$, and surface $= 10 \times 31.416 =$ 314.16. *Ans.* 314.16 square feet.

It is also proved in Geometry that to find the volume of a sphere it is necessary to multiply the surface by $\frac{1}{6}$ the diameter.

Problem.—To find the volume of the sphere of which the radius is 5 feet. The surface was found to be 314.16 feet. $314.16 \times 10 \times \frac{1}{6} = 523.6$ cubic feet. *Ans.*

It is evident in the case of a parallelopipedon of which the width is uniform, length and depth also uniform, that the units of cubic contents are found by multiplying together the number of units in length, breadth, and depth.

Also, in a similar way, in the case of a prism or cylinder, the volume would be found by multiplying the area of the base by the number of units in length or altitude.

And it is proved in Geometry that the volume of a pyramid or cone is $\frac{1}{3}$ of the volume of a prism or cylinder having the same base and equal altitude or length.

And it is proven that the volume of any frustum of

a pyramid or cone is equal to the sum of the volumes of three pyramids or cones, each having the same altitude, the base of one being equal to the lower base of the frustum, the base of another being equal to the upper base of the frustum, and the base of the third being a mean proportional between these two:

A mean proportional between two numbers is expressed by the square root of the product of the two numbers.

Thus, suppose the frustum of a pyramid to have a lower base of 9 square feet and an upper base of 6 square feet, then the mean proportional between these would be $\sqrt{9 \times 4} = \sqrt{36} = 6$. Let the altitude of the frustum be 6 feet, then the sum of the volumes of the three pyramids would be $\frac{1}{3} \times (9+4+6) \times 6 = 38$ cubic feet.

Problem.—The frustum of a cone is 18 feet in height. The lower base contains 25 square feet, the upper base contains 16 square feet. Required to find the volume. *Ans.* 366 cubic feet.

It was found in the case of surfaces, whatever the form of the figure, the area was expressed as some product of two factors, one of length and the other of breadth, whence it followed that in the area of similar surfaces the ratio of the corresponding sides would be squared, because it must enter twice as a factor. Thus in doubling or tripling the sides of any given field the area would be increased to four times, or to nine times the given area, and so on. In the case of volumes

the cubic contents are expressed as the product of three factors, viz.: length, breadth, and depth or thickness.

Hence, in doubling or multiplying by any ratio the dimensions of a given solid to form a similar solid, the ratio, whether two or more than two, must appear three times; that is, must be cubed. Hence, *the volumes of similar solids are to each other as the cubes of their like dimensions.*

All spheres are similar to each other; all cubes are similar to each other; and whatever the form of any prism or cylinder, pyramid or cone, another similar one may be formed.

PROBLEMS.

(1). Required to find the lateral surface of a right pyramid, each side of the triangular base of which measures 4 feet, and the slant height of which is 6 feet. Required also to find the entire surface of the pyramid and the altitude of the pyramid.

Ans. Lateral surface is 36 square feet; entire surface, 42.934 square feet; altitude, 5.89 nearly.

(2). Required the convex surface, the entire surface, and the volume of a cone of which the slant height is 13 feet and the radius of the base of which is 5 feet.

Ans. Convex surface is 204.204 square feet; entire surface is 282.744 square feet; volume is 314.16 cubic feet.

(3). Required to find the convex surface and the entire surface; also volume of a cylinder of which the length or altitude is 10 feet and the radius of the base

of which is 5 inches. Then to find the surface and volume of a similar cylinder, the radius of the base of which is 10 inches.

(4). Required the cubic contents of a box of which the interior dimensions are respectively 2, 3, and 4 feet. Also a similar box of which the least dimension is 10 feet.

(5). Required to find the dimensions of a box of which the volume is 24000 cubic feet and of which the edges are in the ratio of 2, 3, 4. Also a similar box of which the volume is 192000 cubic feet.

In the selection of the following list of problems the author has sought to avoid ambiguity or indefiniteness in the enunciations, at the same time to discard questions of the nature of puzzles and riddles, as unworthy of the serious attention of the arithmetical student. It happens in some instances that statements are included in the enunciations which have no bearing on the solution of the problems, and if the student is thereby misled, he shows a lack of skill in failing to discern what is, and what is not, essential to the solution.

In any case, the student should carefully consider the statement of a problem, to be sure that he understands the given data and the nature of the result required.

Should one meet in the following pages with any term not previously explained, he may consult the vocabulary in the Appendix at the close of the volume.

(1). A boy has gathered 3 pk. 5 qt. 1 pt. of chestnuts. What should he obtain for them at the rate of \$2.56 per bushel? *Ans.* \$2.36.

(2). One boy has 20 cents more than another; both together have 90 cents. How much has each?

Ans. 35 cents, 55 cents.

(3). Two boys sell together some hazel-nuts at the rate of \$3.20 per bushel, and receive \$7.25. One had a bushel and a pint of hazel-nuts more than the other. How should the money be shared between them?

Ans. \$2.00 and \$5.25 respectively.

(4). A pile of wood measures 4 feet in width, 8 feet 6 inches in length, 6 feet 5 inches in height. How many cubic feet does it contain? How many cords? How much is it worth at \$5.50 per cord?

Ans. $218\frac{1}{6}$ cu. ft., or $1\frac{541}{768}$ cd., and is worth \$9.37+.

(5). Another pile of wood is $13\frac{1}{2}$ feet in length, 4 feet wide, $5\frac{1}{4}$ feet high. What is it worth at \$6.40 per cord? *Ans.* \$14.17+.

(6). A load of wood 16 feet 8 inches long, 4 feet wide, 5 feet 3 inches high, is sold for \$21. What is the price per cord? *Ans.* \$7.68.

(7). A pile of wood is 1.2 meters wide, 3.5 meters long, 2 meters high. How many steres does it contain? How much is it worth at \$5 per cord?

Ans. \$11.59 nearly.

(8). A pile of wood is 1 meter wide, 10 meters long, 2 meters high. How much is it worth at \$8.25 per cord.

(9). A cistern measures $4\frac{1}{2}$ feet in width, $3\frac{1}{3}$ feet in

height, and 10 feet in length. How many gallons does it contain? How many liters?

Ans. 1122.1 gallons nearly; 4247.5 liters nearly.

(10). A cistern measures 25 feet in length, 6 feet in width, $5\frac{1}{2}$ feet in depth. How many bushels does it hold? How many hectoliters?

(11). At 5 cents a pound how many kilograms may be bought for $25?

(12). At $5 a gallon how many liters of wine may be bought for $12.75?

(13). An apothecary bought 10 pounds of quinine at $40 per pound, which he used in making 2-grain pills. These he sold at 30 cents per dozen. What was the entire profit? *Ans.* $475.

(14). What is the cost of 118 rods $8\frac{1}{2}$ feet of fencing at $160 per mile?

(15). What should $1\frac{1}{2}$ hectares of land cost if 10 A. 100 sq. rd. cost $850?

(16). The longitude of Philadelphia being 75° 10′ west from Greenwich, what is the difference of time?

(17). If Greenwich time is 5 h. 8 m. 12 s. later than Washington time, and if Lawrence, Kansas, is 95° 15′ west from Greenwich, what is the difference between Washington and Lawrence time?

Ans. Lawrence time is 1 h. 12 m. 48 s. earlier than Washington.

(18). If Rome is 12° 29′ 2″ east from Greenwich, and Fort Leavenworth is 94° 55′ west, what is the difference of time between Fort Leavenworth and Rome?

(19). If St. Petersburg is 30° 18′ 22″ east from Greenwich, and Rome is 89° 32′ 2″ east from Wash-

ington, and Washington 77° 3′ west from Greenwich, when it is midnight at St. Petersburg what is the time at Rome?

(20). What does it cost to carpet a room 21 feet long, 18 feet wide, with carpeting $\frac{3}{4}$ of a yard wide at \$1.625 per yard? *Ans.* \$91.

(21). 12 acres 2 roods is 10% of what quantity?

(22). 24.6 liters is how many per cent of 8.2 hectoliters?

(23). 10 liters is how many per cent of 10 gallons?

(24). \$60 is how many per cent of \$24? How many per cent of 24 cents?

(25). A house costs \$21,500, which is $7\frac{1}{2}$% more than was estimated. What was the estimated cost?

(26). I paid \$300 for a horse, and lost $37\frac{1}{2}$% in selling him. What was the selling price?

(27). A grocer asks 20% more than the cost, but deducts 5% of his asking price. How much profit does he make on goods that cost \$480? *Ans.* \$67.20.

(28). A jeweler sold a bill of goods which cost \$125 at a profit of 50%, but finally collected only 75% of the amount of the sale. How much was the profit finally realized?

(29). Find the simple interest of \$14.50 for 5 months and 16 days at 6% per annum? *Ans.* \$0.401$\frac{1}{6}$.

(30). Find the amount of \$175, at interest for 8 months and 8 days, at 8% per annum. *Ans.* \$184.64$\frac{4}{9}$.

(31). What is the difference between discounting a bill of \$2000, at 10%, and then discounting the remainder at 10% for cash payment, and discounting the whole bill at 20%?

(32). 20% less than \$80 is how many per cent greater than \$60?

(33). 50% of \$1 is how many per cent of 50% of \$2?

(34). A gains 20% on his investment, and then loses 20% of the whole amount. How many per cent does he gain or lose in the whole affair?

(35). A merchant buys goods at a discount of 50% and then sells at full price. How many per cent does he gain?

(36). One-third and a half a third of 10 is how many per cent of one-third and a half a third of 20?

(37). A dealer bought 200 books for \$400. For how much must he offer to sell them in order that he may discount 5% and still make a profit of 20%?

(38). A dealer sells two horses for \$150 each. On one he gains $33\frac{1}{3}$% and on the other loses $33\frac{1}{3}$%. Does he gain or lose in the whole transaction, and how much? How many per cent?

(39). A merchant marks cloth that cost \$2 per yard, so that he may deduct 5% for cash payment, and yet make a profit of 25%. What is the marked price?

(40). A merchant sells goods for \$2 per yard, but wishes to "mark up" the price, so that he may deduct 20% and yet receive the same as before. What must be the advanced price?

(41). I discount 20% on goods marked \$2.50 per yard, and yet make a profit of 25%. What was the first cost?

(42). An agent sells a lot of cotton on a commission of 5% and invests the proceeds in cloth, after deducting the commission for buying, which is also 5%. His commissions for selling and buying amount to \$150.

What were the net proceeds of the sale of cotton? How much was invested in cloth? What was the amount of each commission?

Ans. Net proceeds of sale of cotton, \$1496.25; \$1425 invested in cloth; commission for selling cotton, \$78.75; commission for buying cloth, \$71.25.

(43). I sent a broker \$10,000, with which he bought railroad stock at 90%. He afterwards sold it at an advance of 10% on what he paid for it. Allowing the broker ½% for buying and for selling, what was the amount of his commissions and what was the amount of my profit.

Ans. Amount of commissions, \$55.40 +; profit, \$941.83 nearly.

(44). Bought 450 shares San Miguel Gold Mine stock at 110%; 80 shares La Plata Gold Mine stock at 80%; and exchanged at the same rates for Big Casino stock at 110%, which I sold at 115%. What was the whole gain and the rate per cent of gain?

(45). What will it cost to insure my house to the amount of \$2500 for two years, at a premium of 1½%, the policy and survey costing \$2?

(46). What will it cost to obtain an insurance of \$50,000, and enough more to include the cost of insurance, on a store and the goods, the goods being insured for two-thirds as much as the store, the rate for the store being 2% and the goods ½%, each policy and survey costing \$1.50? In the case of fire and the total destruction of the property how much would the insurance company lose, after deducting what was previously received?

Ans. Cost of insurance, \$715.77+ ; lost by the insurance company, \$50,000.

(47). A merchant insures his goods at the rate of $\frac{1}{4}$%, to cover three-fourths of their value, that is to include the cost of insurance. The whole expense was \$502, including the policy, which cost \$2. What was the value of the goods? In the case of fire how much would the merchant lose, including the cost of insurance, and how much would the company lose?

(48). If the City of Lawrence requires \$10,000 for the purpose of boring for coal, this sum to be obtained by tax, and if the property is valued at \$2,000,000, to be assessed at $\frac{3}{4}$ its value, and a commission of 5% be allowed for the expense of collecting, what will be the tax of a man whose property is valued at \$2380?

(49). What is the present worth of a note of \$2000 which matures in 93 days, without interest, money being reckoned worth 10% per annum? What is the difference between the present worth and the net proceeds, if discounted at the bank? Suppose the same note drawing interest and apply the same questions.

(50). I make a profit of 15% in selling goods at 20% less than the marked price. What would be the profit in selling at the marked price?

(51). What is the face of a note drawn for 93 days, discounted at the bank at 10%, of which the proceeds are \$5000. What is the rate of interest paid on net proceeds?

(52). A note for \$1000, dated April 12, 1877, due, with 12% interest, in 1 year, 6 months, and 15 days (without grace), was discounted at the bank Sept. 18,

1877. On this note, payments were indorsed as follows: \$150, July 4; \$300, Aug. 12. What was received on this note, discounting at 12%, and reckoning 30 days to the month in all cases?

(53). What is the cost of a draft for \$2000, payable 30 days after sight, legal interest being 7%, and the price of exchange at 4% premium? What would it cost if payable at sight?

(54). Which is better, to invest \$50,000 in 6% gold interest bearing bonds, gold being at a premium of 20%, or to loan at 10% on good security? At what price must the bonds be bought to realize 8% on the investment?

(55). Gold being worth 110%, which is more profitable, to sell wheat at \$2 in currency or \$1.80 in gold? How much?

(56). A tree 5 rods from the east bank of a lake is exactly east of a tree 10 rods from the west bank of the lake. It is 300 rods from the first tree to a third tree exactly north of it, and 500 rods from that to the second tree. What is the width of the lake?

Ans. 385 rods.

(57). The door of a mill is 6 feet high. How wide must this door be in order that a circular saw, 8 feet in diameter, may be taken through it?

(58). The stock of a railroad company being valued at \$2,575,000, the expenses of a quarter amount to \$31,800, the gross receipts \$85,267.50, and the company declare a dividend of 2%. Required to find what surplus funds will remain on hand. *Ans.* \$1967.50.

(59). A man was receiving \$500 annual rent from his farm, but finally sold it for \$8500, and invested in

bonds bearing 6% interest, paying at the rate of 104%, and a brokerage of $\frac{1}{2}$%. What was the annual loss resulting from the transaction? *Ans.* $11.96.

(60). A man sells $5000 of U. S. bonds bearing 6% interest at 105$\frac{3}{4}$%, allowing $\frac{1}{4}$% brokerage, and invests the proceeds in a farm on which he pays taxes at the rate of $\frac{7}{8}$% on $\frac{3}{4}$ its actual value. He receives an annual rent of $380. How much has he gained or lost by the transactions?

(61). The assets of a bank were $15,760, and a settlement was made by paying 12$\frac{1}{2}$ cents on a dollar. What was the amount of liabilities?

(62). A merchant bought 50 gallons of alcohol at 90 cents a gallon. By adding a certain amount of water and selling the mixture at $1 per gallon, he gained 25%. How much water was added?

(63). A grocer bought a hogshead of molasses on credit, agreeing to pay 10% annual interest on the cash price until settled. At the end of 8 months he sold the molasses for $60 and settled his indebtedness, when he found he had gained 25%. What was the cash price? *Ans.* $45.

(64). A school district having to build a school-house, issued bonds for $1000, bearing 10% annual interest, due in 10 years. The bonds were sold at a discount of 10%, and a school-house built with the proceeds. The sum of $25 was expended each year for repairs, and at the end of 10 years the school-house was totally destroyed by fire without insurance. Required to find the average annual expense incurred by the district on account of the school-house. *Ans.* $225.

(65). The hour and minute hands of a watch being exactly together at 12 o'clock, how many times will they be together in the next 12 hours, and what is the interval between two successive conjunctions of the hands?

(66). At a certain instant of time, the hour hand of a watch pointed between the figures 2 and 3 on the face of the dial, at the same time the minute hand pointed between the figures 3 and 4. Within an hour the hands had exactly changed places. Required, the time of the first observation.

(67). A man owes a debt to be paid in 4 equal installments, at 4, 8, 12, and 16 months respectively without interest. Allowing discount at 6% per annum, he finds that $1000 will cancel the indebtedness at once. How much does he owe?

(68). A and B can do a certain work in $2\frac{1}{2}$ days; A and C in $3\frac{1}{3}$ days; and B and C in $4\frac{1}{4}$ days. Required the time in which A, B, and C, working together, can do the same work, and the time required for each to do it alone.

Ans. All together in $2\frac{22}{159}$ days; A in $4\frac{24}{79}$ days; B in $5\frac{55}{57}$ days; and C in $14\frac{18}{23}$ days.

(69). $\frac{3}{5}$ of A's age equals $\frac{6}{25}$ of B's. If each were 10 years older, $\frac{1}{2}$ of A's age would then equal $\frac{1}{4}$ of B's. Required their ages.

[If $\frac{3}{5}$ of A's age equals $\frac{6}{25}$ of B's age, then $\frac{1}{5}$ of A's age equals $\frac{2}{25}$ of B's age, or A's age equals $\frac{2}{5}$ of B's age. In 10 years A's age would be half of B's; that is, (10 years + $\frac{2}{5}$ of B's age) would equal $\frac{1}{2}$ of (B's age + 10 years), or ($\frac{4}{5}$ of B's age + 20 years) = (B's age + 10

years). Hence $\frac{1}{5}$ of B's age $=$ 10 years, and B's age is 50 years, and A's age is $\frac{2}{5}$ of 50 years, that is 20 years. *Ans.*]

(70). The sum of three numbers is 207; $\frac{1}{2}$ of the first, $\frac{1}{3}$ of the second, and $\frac{1}{4}$ of the third are equal to each other. What are the three numbers?

Ans. 46, 69, 92.

(71). A room is 15 feet wide, 20 feet long, 18¾ feet high. What is the length of a line extending from the upper N. W. corner to the lower S. E. corner of the room?

(72). A father is 40 years of age and his son 10 years. When will the father's age be only twice that of the son?

(73). A boy bought 90 apples at the rate of 2 for a cent, and 90 more at the rate of 3 for a cent. He sold them all at the rate of 5 for 2 cents. How much did he gain or lose? What rate per cent? *Ans.* to last, 4%.

(74). A planter sold a bale of cotton for \$50, then bought it back for \$40, and finally sold it again for \$65. How much did he gain? *Ans.* \$25.

[NOTE.—This problem has received many different solutions from skillful accountants, but a little consideration should suffice to remove all disagreement. What is profit? Profit is "the excess of the selling price above the buying price," or it is the excess of the net proceeds of sale above the cost. If this be granted, there can scarcely be a difference of opinion in regard to the solution of the given question. We must know first *the cost*, then *the proceeds of sale.* In the present case the original cost of the bale is not stated, and in selling for \$50, it is impossible to tell whether there was or was not any profit in that part of the transaction. Buying back for \$40 is the beginning of another transaction which

may end in profit or loss. If the bale should be burned up before a sale were effected, there would of course be a loss, as indeed might happen in many ways. But in this case the bale is again sold for $65, and the profit is $65 – $40 = $25.]

(75). A grocer has sugars worth respectively 8, 9, 12, and 13 cents per pound. He wishes to make a mixture of 1000 pounds, worth 10 cents per pound. Required to know what amounts of each may be used.

Ans. 375, 250, 125, and 250 pounds respectively; or 250, 375, 250, and 125 respectively; or in many other proportions.

(76). A trader sold 50 horses at $100 each. He gained 10% on 20 of these, lost 20% on 20 others, and gained 20% on the balance. How much did he lose?

Ans. $151.51+.

(77). A merchant sells corn at a profit of 20 cents on a bushel, and gains at the rate of 25%. How much must he advance on this price in order to gain 40% on first cost? *Ans.* 12 cents on a bushel.

(78). Find the square roots of the numbers in the following list, correct to the third decimal place, 325.325, 1.5625, 40.96, 2000, 6.25.

Ans. to first 18.036+; to the last 2.5.

(79). A laborer received $1 for each day that he worked, and paid 25 cents for each day that he was idle. At the end of 24 days he received $11.50. How many days was he idle? *Ans.* 10 days.

(80). Required to find the least number which being divided by 6, or by 7, or by 8, or by 9, leaves in each case a remainder of 4. *Ans.* 508.

(81). Hiero, King of Syracuse, gave his goldsmith

14 pounds of gold and $3\frac{1}{2}$ pounds of silver, out of which to make a crown. A cubic inch of gold weighs $19\frac{1}{4}$ times, and silver $10\frac{1}{2}$ times as much as a cubic inch of water, and the crown was found to weigh $14\frac{5}{8}$ times as much as water. Suspecting that some of the gold had been replaced by silver, Hiero requested Archimedes to find how much. What was the fact as indicated by the above statement?

Ans. $3\frac{1}{78}$ pounds of gold had been replaced by silver.

[**NOTE.—First find how much the specific gravity of the crown should have been, that is, how many times heavier than water, then find what would be the effect of changing 1 pound of gold for 1 pound of silver, and find how many pounds would be required in exchange to make the actual difference.**]

(82). Find the sum of $\frac{5}{9}$ of 4 yards 2 feet, $\frac{9}{13}$ of 3 feet 9 inches, $\frac{3}{11}$ of 2 yards 2 feet 2 inches.

Ans. 4 yards $7\frac{92}{429}$ inches.

(83). Reduce to lowest terms $\frac{3591}{3961}$ and $\frac{5459}{5831}$.

(84). Simplify $\dfrac{\frac{1}{4}+\frac{3}{13}+\frac{8}{24}-\frac{1}{4}\times\frac{3}{13}\times\frac{8}{24}}{1-\frac{1}{4}\times\frac{3}{13}-\frac{3}{13}\times\frac{8}{24}-\frac{8}{24}\times\frac{1}{4}}$.

Ans. $1\frac{1}{61}$.

(85). What decimal part of £1 is $\frac{2}{3}$ of 7s. $9\frac{1}{2}$d.

Ans. £0.2597$\frac{2}{3}$.

(86). Reduce $\frac{37\frac{1}{2}}{1893\frac{3}{4}} \div \frac{173\frac{3}{4}}{669\frac{6}{7}}$. *Ans.* $\frac{54}{707}$.

(87). Supposing gold worth \$19.20 per ounce, and silver worth \$1.20 per ounce, what is the value of 3 lb. 2 oz. 15 pwt. 15 gr. of metal of which $\frac{22}{25}$ are pure gold and the balance silver? *Ans.* \$660.83$\frac{1}{4}$.

(88). If £1 = 24.55 francs, and 8.24 francs = \$1.65, what is the value in dollars of £350 17s. 6d.?

Ans. \$1724.887 +.

(89). How many acres in a rectangular field of which the length is $1\frac{7}{8}$ times the width, and around which a person can walk in $51\frac{3}{4}$ minutes, walking at the rate of $3\frac{3}{11}$ miles per hour.

Ans. 289 A. $30\frac{50}{121}$ sq. rd.

(90). A and B run a race. B starts 10 rods in advance of A, who is 100 yards from the end of the course, and both reach the end together. If B were 100 rods from the end of the course, how far must A be behind so that both shall reach the end together?

(91). A bought a house for $4000, but sold it to B at a certain per cent of loss. B sold it again to C at the same per cent of loss, receiving $3240. Required to find A's and B's per cent of loss.

(92). A square field is inclosed by a fence 5 rails high, the rails $16\frac{1}{2}$ feet in length. The number of acres is $\frac{1}{20}$ the number of rails. Required to find the number of acres.

(93). Required to find the diameter of the base of a cylindrical measure which is five inches deep, to contain exactly a liquid quart. *Ans.* 3.82 inches nearly.

(94). A man paid $100 for a square piece of land, each side of which measured 100 feet. At the same rate, what should be the length of one side of a square piece of land that could be bought for $625?

Ans. 250 feet.

(95). If it costs $15.75 to gild a globe 25 inches in diameter, how much at the same rate should it cost to gild a globe 10 inches in diameter? *Ans.* $2.52.

(96). A can do a piece of work in 12 days, B can do the same in 16 days. They work together 4 days,

when A leaves, and engages C to take his place, who can do only $\frac{1}{4}$ the work of A. Required to know when the work will be completed.

(97). Required to find the capacity in gallons of a cylindrical tank, of which the interior diameter is $3\frac{1}{2}$ feet and the length 10 feet.

(98). A cylindrical tank, whose interior diameter is 2 feet and length 3 feet, weighs 650 pounds. Required to find the total weight of tank and water, when the tank is full of distilled water.

(99). What is the weight of a marble monument in the form of a square pyramid, 10 feet in height, each side of the base measuring 4 feet, and each cubic inch of marble weighing $2\frac{1}{4}$ times as much as the same volume of distilled water?

(100). A and B send packages by express a distance of 80 miles. B's package weighs only $\frac{3}{4}$ as much as A's. The charge per mile for the last 30 miles is only $\frac{3}{4}$ as much as for the first 50 miles. The total expense is $4.48. Required to find the amount paid by each.

APPENDIX.

APPENDIX.

Containing Various Matters which may be Useful for Reference.

I.—*Amount of $1, at Compound Interest, from One to Thirty Years.*

YEARS.	3 per cent.	3½ per cent.	4 per cent.	4½ per cent.	5 per cent.	6 per cent.	7 per cent.
1	1.0300 0000	1.0350 0000	1.0400 0000	1.0450 0000	1.0500 000	1.0600 000	1.0700 000
2	1.0609 0000	1.0712 2500	1.0816 0000	1.0920 2500	1.1025 000	1.1236 000	1.1449 000
3	1.0927 2700	1.1087 1787	1.1248 6400	1.1411 6612	1.1576 250	1.1910 160	1 2250 430
4	1.1255 0881	1.1475 2300	1.1698 5856	1.1925 1860	1.2155 063	1.2624 770	1.3107 960
5	1.1592 7407	1.1876 8631	1.2166 5290	1.2461 8194	1.2762 816	1.3382 256	1.4025 517
6	1.1940 5230	1.2292 5533	1.2653 1902	1.3022 6012	1.3400 956	1.4185 191	1.5007 304
7	1.2298 7387	1.2722 7926	1.3159 3178	1.3608 6183	1.4071 004	1.5036 303	1.6057 815
8	1.2667 7008	1.3168 0904	1.3685 6905	1.4221 0061	1.4774 554	1.5938 481	1.7181 862
9	1.3047 7318	1.3628 9735	1.4233 1181	1.4860 9514	1.5513 282	1.6894 790	1.8384 592
10	1.3439 1638	1.4105 9876	1.4802 4428	1.5529 6942	1.6288 946	1.7908 477	1.9671 514
11	1.3842 3387	1.4599 6972	1.5394 5406	1.6228 5305	1.7103 394	1.8982 986	2.1048 520
12	1.4257 6089	1.5110 6866	1.6010 3222	1.6958 8143	1.7958 563	2.0121 965	2.2521 916
13	1.4685 3371	1.5639 5606	1.6650 7351	1.7721 9610	1.8856 491	2.1329 283	2.4098 450
14	1.5125 8972	1.6186 9452	1.7316 7645	1.8519 4492	1.9799 316	2.2609 040	2.5785 342
15	1.5579 6742	1.6753 4883	1.8009 4351	1.9352 8244	2.0789 282	2.3965 582	2.7590 315
16	1.6047 0644	1.7339 8604	1.8729 8125	2.0223 7015	2.1828 746	2.5403 517	2.9521 638
17	1.6528 4763	1.7946 7555	1.9479 0050	2.1133 7681	2.2920 183	2.6927 728	3.1588 152
18	1.7024 3306	1.8574 8920	2.0258 1652	2.2084 7877	2.4066 192	2.8543 392	3.3799 323
19	1.7535 0605	1.9225 0132	2.1068 4918	2.3078 6031	2.5269 502	3.0255 995	3.6165 275
20	1.8061 1123	1.9897 8886	2.1911 2314	2.4117 1402	2.6532 977	3.2071 355	3.8696 845
21	1.8602 9457	2.0594 3147	2.2787 6807	2.5202 4116	2.7859 626	3.3995 636	4.1405 624
22	1.9161 0341	2.1315 1158	2.3699 1879	2.6336 5201	2.9252 607	3.6035 374	4.4304 017
23	1.9735 8651	2.2061 1448	2.4647 1555	2.7521 6635	3.0715 238	3.8197 497	4.7405 299
24	2.0327 9411	2.2833 2849	2.5633 0417	2.8760 1383	3.[illegible]50 999	4.0489 346	5.0723 670
25	2.0937 7793	2.3632 4498	2.6658 3633	3.0054 3446	[illegible]63 549	4.2918 707	5.4274 326
26	2.1565 9327	2.4459 5856	2.7724 6979	3.1406 7901	[illegible]56 727	4.5493 830	5.8073 529
27	2.2212 8901	2.5315 6711	2.8833 6858	3.2820 0956	[illegible]34 563	4.8223 459	6.2138 676
28	2.2879 2768	2.6201 7196	2.9987 0332	3.4296 9999	3.9201 291	5.1116 867	6.6488 384
29	2.3565 6551	2.7118 7798	3.1186 5145	3.5840 3649	4.1161 356	5.4183 879	7.1142 571
30	2.4272 6247	2.8067 9370	3.2433 9751	3.7453 1813	4.3219 424	5.7434 912	7.6122 550

II.

***DISCOUNT TABLE**, showing the present value of an Annuity of $1 per annum for a series of years.*

YEARS.	4 per cent.	5 per cent.	6 per cent.	7 per cent.	8 per cent.
1	.961538	.952381	.943396	.934579	.925926
2	1.886095	1.859410	1.833393	1.808018	1.783265
3	2.775091	2.723248	2.673012	2.624316	2.577097
4	3.629895	3.545951	3.465106	3.387211	3.312127
5	4.451822	4.329477	4.212364	4.100197	3.992710
6	5.242137	5.075692	4.917324	4.766540	4.622880
7	6.002055	5.786373	5.582381	5.389289	5.206370
8	6.732745	6.463213	6.209794	5.971299	5.746639
9	7.435332	7.107822	6.801692	6.515232	6.246888
10	8.110896	7.721735	7.360087	7.023582	6.710081
11	8.760477	8.306414	7.886875	7.498675	7.138964
12	9.385074	8.863252	8.383844	7.942686	7.536078
13	9.985648	9.393573	8.852683	8.357651	7.903776
14	10.563123	9.898641	9.294984	8.745468	8.244237
15	11.118387	10.379658	9.712249	9.107914	8.559479
16	11.652296	10.837770	10.105895	9.446649	8.851369
17	12.165669	11.274066	10.477260	9.763223	9.121638
18	12.659297	11.689587	10.827603	10.059087	9.371887
19	13.133939	12.085321	11.158116	10.335595	9.603599
20	13.590326	12.462210	11.469921	10.594014	9.818147
21	14.029160	12.821153	11.764077	10.835527	10.016803
22	14.451115	13.163003	12.041582	11.061241	10.200744
23	14.856842	13.488574	12.303379	11.272187	10.371059
24	15.246963	13.798642	12.550358	11.469334	10.528758
25	15.622080	14.093945	12.783356	11.653583	10.674776
26	15.982769	14.375185	13.003166	11.825779	10.809978
27	16.329586	14.643034	13.210534	11.986709	10.935165
28	16.663063	14.898127	13.406164	12.137111	11.051078
29	16.983715	15.141074	13.590721	12.277674	11.158406
30	17.292033	15.372451	13.764831	12.409041	11.257783
31	17.588494	15.592811	13.929086	12.531814	11.349799
32	17.873552	15.802677	14.084043	12.646555	11.434999
33	18.147646	16.002549	14.230230	12.753790	11.513888
34	18.411198	16.192904	14.368141	12.854009	11.586934
35	18.664613	16.374194	14.498246	12.947672	11.654568
36	18.908282	16.546852	14.620987	13.035208	11.717193
37	19.142579	16.711287	14.736780	13.117017	11.775179
38	19.367864	16.867893	14.846019	13.193473	11.828869
39	19.584485	17.017041	14.949075	13.264928	11.878582
40	19.792774	17.159086	15.046297	13.331709	11.924613
41	19.993052	17.294368	15.138016	13.394120	11.967235
42	20.185627	17.423208	15.224543	13.452449	12.006699
43	20.370795	17.545912	15.306173	13.506962	12.043240
44	20.548841	17.662773	15.383182	13.557908	12.077074
45	20.720040	17.774070	15.455832	13.605522	12.108402
46	20.884654	17.880067	15.524370	13.650020	12.137409
47	21.042936	17.981016	15.589028	13.691608	12.164267
48	21.195131	18.077158	15.650027	13.730474	12.189136
49	21.341472	18.168722	15.707572	13.766799	12.212163
50	21.482185	18.255925	15.761861	13.800746	12.333485

II. (c.)

From Tables like the foregoing are derived a great variety of tables in use in the business of Life Insurance. The following are selected from a large number, as used by the Washington Life Insurance Company.

ORDINARY LIFE TABLE. *Annual, Semi-Annual, and Quarterly Premiums for an Insurance of One Thousand Dollars, on a Single Life, Payable at Death.*

Age	Annual Premium.	Semi-Annual Premium.	Quarterly Premium.	Age.
25	19 89	10 25	5 28	25
26	20 40	10 61	5 41	26
27	20 93	10 89	5 55	27
28	21 48	11 17	5 70	28
29	22 07	11 48	5 85	29
30	22 70	11 80	6 01	30
31	23 25	12 14	6 19	31
32	24 05	12 50	6 37	32
33	24 78	12 89	6 57	33
34	25 56	13 29	6 77	34
35	26 38	13 72	6 99	35
36	27 25	14 17	7 22	36
37	28 17	14 65	7 47	37
38	29 15	15 16	7 73	38
39	30 19	15 70	8 00	39
40	31 30	16 28	8 29	40
41	32 47	16 89	8 60	41
42	33 72	17 54	8 94	42
43	35 05	18 23	9 29	43
44	36 46	18 96	9 66	44
45	37 97	19 75	10 06	45
46	39 58	20 58	10 49	46
47	41 30	21 48	10 95	47
48	43 13	22 43	11 43	48
49	45 09	23 45	11 95	49
50	47 18	24 54	12 50	50
51	49 40	25 69	13 10	51
52	51 78	26 93	13 72	52
53	54 31	28 24	14 39	53
54	57 02	29 65	15 11	54
55	59 91	31 16	15 88	55
56	63 00	32 76	16 70	56
57	66 29	34 47	17 57	57
58	69 82	36 30	18 50	58
59	73 60	38 27	19 50	59
60	77 63	40 37	20 57	60
61	81 96	42 62	21 72	61
62	86 58	45 02	22 94	62
63	91 54	47 60	24 26	63
64	96 86	50 37	25 67	64
65	102 55	53 33	27 18	65

III. (d.)

CHILDREN'S ENDOWMENT TABLE. Annual Premiums charged to secure Endowments of $1,000.

Present Age.	Payable at the age of 18.		Payable at the age of 21.		Payable at the age of 25.	
	No Premiums returned in case of previous death.	All Premiums returned in case of previous death.	No Premiums returned in case of previous death.	All Premiums returned in case of previous death.	No Premiums returned in case of previous death.	All Premiums return'd in case of previous death.
1	$40.38	$46.78	$31.80	$37.16	$23.85	$28.21
2	44.77	50.30	34.90	39.64	25.93	29.89
3	49.52	54.44	38.18	42.52	28.09	31.81
4	54.75	59.32	41.71	45.86	30.36	33.99
5	60.68	65.06	45.60	49.67	32.82	36.44
6	67.50	71.85	49.97	54.07	35.50	39.18
7	75.52	79.94	54.93	59.13	38.47	42.27
8	85.13	89.69	60.65	65 01	41.79	45.74
9	96.87	101.66	67.33	71.90	45.54	49.66
10	111.58	116.66	75.24	80.07	49.80	54.13
11	130.53	135.96	84.78	89.88	54.70	59.24
12	155.88	161.71	96.49	101.89	60.39	65.15
13	191.45	197.78	111.18	116.91	67.06	72.06
14	244.93	251.89	130.13	136.22	74.99	80.22
15	......	......	155.49	161.97	84.55	90.03
16	......	......	191.10	198.02	96.30	102.01
17	......	......	244.64	252.09	111.03	117.00
18	......	......	......	......	130.03	136.29

III. (*e.*)—***SINGLE PREMIUMS to secure $1,000, payable as indicated, or at death if prior.***

WITH PROFITS.

Age	At Death only.	In 35 years.	In 30 years.	In 25 years.	In 20 years.	In 15 years.	In 10 years.	Age
25	326 58	407 60	453 58	514 70	594 16	696 27	826 64	25
26	332 58	409 54	454 92	515 58	594 72	696 61	826 83	26
27	338 83	411 69	456 39	516 55	595 34	696 99	827 03	27
28	345 31	414 05	458 01	517 61	596 01	697 38	827 26	28
29	352 05	416 65	459 79	518 77	596 74	697 81	827 48	29
30	359 05	419 51	461 76	520 06	597 54	698 28	827 74	30
31	366 33	422 66	463 94	521 48	598 42	698 79	828 01	31
32	373 89	426 11	466 34	523 05	599 37	699 34	828 20	32
33	381 73	429 90	468 99	524 78	600 43	699 94	828 60	33
34	389 88	434 06	471 92	526 70	601 60	700 61	828 95	34
35	398 34	438 63	475 16	528 83	602 90	701 33	829 32	35
36	407 11	443 60	478 71	531 19	604 [illegible]3	702 12	829 71	36
37	416 21	449 03	482 64	533 80	605 92	703 00	830 15	37
38	425 64	454 94	486 95	536 69	607 69	703 97	830 62	38
39	435 42	461 36	491 69	539 90	609 67	705 06	831 15	39
40	445 55	468 31	496 89	543 45	611 86	706 26	831 72	40
41	456 04	475 81	502 58	547 37	614 31	707 61	832 36	41
42	466 89	483 89	508 79	551 70	617 02	709 11	833 06	42
43	478 11	492 55	515 55	556 47	620 04	710 79	833 85	43
44	489 71	501 81	522 89	561 71	623 39	712 67	834 72	44
45	501 69	511 59	530 84	567 46	627 11	714 76	835 70	45
46	514 04	522 16	539 42	573 75	631 22	717 09	836 80	46
47	526 78	5[illegible]3 24	548 63	580 62	635 76	719 70	838 03	47
48	539 88	544 91	558 47	588 07	640 74	722 58	839 39	48
49	553 33	557 15	568 94	596 13	646 20	725 76	840 91	49
50	567 13	569 94	580 05	604 82	652 16	723 28	842 59	50
51	581 24		591 75	614 13	658 64	733 13	844 44	51
52	595 66		604 05	624 06	665 66	737 [illegible]5	846 48	52
53	610 36		616 91	634 61	673 23	741 94	848 71	53
54	625 33		630 32	645 77	681 39	746 95	851 16	54
55	640 54		644 24	657 54	690 13	752 39	853 84	55
56	655 99			669 88	699 46	758 29	856 76	56
57	671 64			682 83	709 40	764 67	859 95	57
58	687 48			696 26	719 92	771 55	863 43	58
59	703 49			710 22	731 03	778 96	867 22	59
60	719 65			724 66	742 70	786 90	871 35	60
61	735 92				754 91	795 37	875 83	61
62	752 26				791 65	804 [illegible]6	880 66	62
63	768 67				780 89	813 89	885 89	63
64	785 10				794 58	823 91	891 53	64
65	801 52				808 67	834 43	897 56	65
66	817 92				823 14	845 43	904 01	66
67	834 25				837 90	856 88	910 87	67
68	850 47				852 88	868 76	918 11	68
69	866 56				868 07	881 02	925 73	69
70	882 48				883 35	893 62	933 71	70

IV.—*TABLE showing the values in United States money of the pure gold or silver, representing respectively the monetary units and standard coins of foreign countries, as announced by the Secretary of the Treasury, January 1, 1878.*

COUNTRY.	Monetary unit.	Standard.	Value in U. S. money.	Standard coin.
Austria	Florin	Silver	.45,3	Florin.
Belgium	Franc	Gold and silver	.19,3	5, 10, and 20 francs.
Bolivia	Dollar	Gold and silver	.96,5	Escudo, ½ bolivar and bolivar.
Brazil	Milreis of 1,000 reis	Gold	.54,5	None.
British Possessions in N. America	Dollar	Gold	$1.00	
Bogota	Peso	Gold	.96,5	
Central America	Dollar	Silver	.91,8	Dollar.
Chili	Peso	Gold	.91,2	Condor, doubloon, and escudo.
Denmark	Crown	Gold	.26,8	10 and 20 crowns.
Ecuador	Dollar	Silver	.91,8	Dollar.
Egypt	Pound of 100 piasters	Gold	4.97,4	5, 10, 25, and 50 piasters.
France	Franc	Gold and silver	.19,3	5, 10, and 20 francs.
Great Britain	Pound sterling	Gold	4.86,6½	½ sovereign and sovereign.
Greece	Drachma	Gold and silver	.19,3	5, 10, 20, 50, and 100 drachmas.
German Empire	Mark	Gold	.23,8	5, 10, and 20 marks.
Japan	Yen	Gold	.99,7	1, 2, 5, 10, and 20 yen.
India	Rupee of 16 annas	Silver	.43,6	
Italy	Lira	Gold and silver	.19,3	5, 10, 20, 50, and 100 lire.
Liberia	Dollar	Gold	1.00	
Mexico	Dollar	Silver	.99,8	Peso or dollar, 5, 10, 25, and 50 centavo.
Netherlands	Florin	Gold and silver	.38,5	Florin; 10 guldens, gold ($4.01.9).
Norway	Crown	Gold	.26,8	10 and 20 crowns.
Peru	Dollar	Silver	.91,8	
Portugal	Milreis of 1,000 reis	Gold	1.08	2, 5, and 10 milreis.
Russia	Rouble of 100 copecks	Silver	.73,4	¼, ½, and 1 rouble.
Sandwich Islands	Dollar	Gold	1.00	
Spain	Peseta of 100 centimes	Gold and silver	.19,3	5, 10, 20, 50, and 100 pesetas.
Sweden	Crown	Gold	.26,8	10 and 20 crowns.
Switzerland	Franc	Gold and silver	.19,3	5, 10, and 20 francs.
Tripoli	Mahbub of 20 piasters	Silver	.82,9	
Tunis	Piaster of 16 caroubs	Silver	.11,8	
Turkey	Piaster	Gold	.04,3	25, 50, 100, 250, and 500 piasters.
United States of Columbia	Peso	Silver	.91,8	

V.

The following is selected from the United States Treasury regulations, regarding moneys receivable for duties:

ART. 1003.—Payments in gold coin should be weighed by single pieces; but if in bulk, the coins must be separately examined and tested as far as necessary; one dollar pieces must be separated from larger coins and weighed apart, and the weighing in bulk must be done by amounts of ten dollars, one hundred dollars, one thousand dollars or multiples thereof.

In weighing coins, the ounce troy and the decimals thereof are to be used.

The standard weight and the least current weight of certain specified sums in gold coins above the dollar are as follows:

Amount.	Standard weight.	Least current weight.
$100	5.375 oz.	5.348 oz.
500	26.875 oz.	26.741 oz.
1000	53.750 oz.	53.481 oz.
5000	268.750 oz.	267.407 oz.

The gold dollar continuing current until the allowed deviation from standard weight in manufacture is exceeded by wear and abrasion, 5,000 pieces will be current when weighing not less than 266$\frac{145}{1000}$ ounces troy.

The standard weight and the least current weight of single gold coins of the United States above the dollar, are as follows:

Coin.	Standard weight.	Least current weight.
Quarter eagle.	64.5 grs.	64.18 grs.
Three dollar.	77.4 grs.	77.02 grs.
Half eagle.	129.0 grs.	128.36 grs.
Eagle.	258.0 grs.	256.71 grs.
Double eagle.	516.0 grs.	513.42 grs.

As the coinage law tolerates a deviation from the standard weight of one-quarter of a grain, or less, in the manufacture of the dollar piece, that coin will be current and receivable so long as it is not reduced below 25$\frac{55}{100}$ grains in actual weight.

[From *Heyl's U. S. Import Duties*, 1877.]

VI.

The specific gravity of a body is its weight as compared with that of an equal bulk of some other body adopted as a standard.

For bodies in the liquid or solid form, water at the temperature of 60° Fahrenheit with the barometer at 30 inches at sea level, is the standard. Gases are compared with air.

Atmospheric air, at 60° Fahrenheit and under the pressure of one atmosphere, weighs $\frac{1}{815}$ part of the same volume of water.

Table of Specific Gravities and Weights of a few important substances.

NAMES OF SUBSTANCES.	Sp. grav	Wt. of cu. ft. in pounds.
Alcohol, pure	.793	49.43
" of commerce	.834	52.1
" proof spirit	.916	57.2
Brick, common	....	125.
" best pressed	....	150.
Elm, dry, average	....	35.
Granite, "	2.72	170.
Gravel and sand	....	100 to 119
Gold, pure, cast	19.258	1204.
" " native	19.32	1206.
Iron, cast, average	7.15	450.
" wrought, average	7.77	485.
Masonry, granite	....	165.
" brickwork	....	140.
Mercury	13.58	846.
Mortar, hard, average	1.65	103.
Oak, dry white, "	.83	51.8
Pine, dry white, "	.40	25.
Silver	10.5	655.
Slate, average	2.8	175.

VII.

Table of Avoirdupois pounds in a bushel, as established by law or custom.

PRODUCE.	WEIGHT.
Barley	48.
Beans	60.
Buckwheat	45.
Clover seeds	60.
Dried apples	28.
Dried peaches	28.
Flax seed	56.
Hemp seed	44.
Indian corn	56.
Indian corn in ear	70.
Indian corn meal	50.
Oats	32.
Onions	52.
Potatoes	60.
Rye	56.
Salt	50.
Wheat	60.
Wheat bran	20.

In a few of the States, slight variations from the above may be found.

Grain, seeds, and small fruits are often sold by the standard bushel measure of 2150.42 cubic inches, in which case the measure must be even full, or *stricken measure;* that is, made even by moving a round stick, which reaches across the measure, from side to side, until any material above the edges of the measure shall be removed.

Coarse vegetables, large fruits, corn in the ear, and other bulky articles, are often sold by *heaped measure.*

A bushel *heaped measure* contains 2747.72 cubic inches, and is supposed to be obtained by heaping the standard bushel measure until a cone is formed 6 inches high, with a base 19½ inches in diameter. This is on the supposition that the sides of the vessel used as a measure are ½ inch in thickness.

A bushel heaped measure is very little more than 5 pecks of stricken measure, but is usually reckoned as equivalent to it.

A quart of heaped measure contains 85.866 cubic inches.

Bricks.

There is no uniformity in the size of bricks made in this country, and no law regulating it. In England the size is fixed by law, $8\frac{3}{4} \times 4\frac{3}{8} \times 2\frac{3}{4}$, containing about $105\frac{1}{4}$ cubic inches. A common size in this country is $8\frac{1}{4} \times 4 \times 2$, containing 66 cubic inches, though often they are much smaller.

Measurement of Hay.

Hay is usually measured by the ton, and weighed on the scales. It is, however, sometimes estimated without weighing, at the rate of 450 cubic feet of fine hay or 550 feet of coarse hay to the ton, well settled in the mow or stack.

An average quality of hay may be reckoned at 500 cubic feet to the ton.

Weight of Nails.

Different makers vary in the sizes and weights of nails of the same name.

The following table shows the ordinary sizes and weights of cut nails.

KIND.	Inches in length.	No. per pound.
3 penny	1	557
4 "	$1\frac{1}{4}$	353
5 "	$1\frac{3}{4}$	232
6 "	2	175
7 "	$2\frac{1}{4}$	141
8 "	$2\frac{1}{2}$	101
10 "	$2\frac{3}{4}$	68
12 "	3	54
20 "	$3\frac{1}{2}$	34

VIII.

Latitude and Longitude of various places, chiefly derived from the American Nautical Almanac. Longitude reckoned from Washington, west.

Place.	Latitude.			Longitude.		
Albany	42°	39′	49″	356°	41′	47″
Allegheny	40°	27′	36″	2°	57′	40″
Ann Arbor	42°	16′	48″	6°	40′	46″
Athens	37°	58′	20″	259°	13′	11″
Berlin	52°	30′	17″	269°	33′	8″
Cambridge, Eng	52°	12′	52″	282°	51′	18″
Cambridge, Mass	42°	22′	48″	354°	4′	43″
Charleston, S. C	32°	46′	44″	2°	52′	38″
Chicago	41°	50′	1″	10°	33′	40″
Cincinnati	39°	6′	26″	7°	26′	44″
Denver	39°	45′	2″	27°	56′	41″
Dubuque	42°	29′	55″	13°	36′	56″
Edinburgh	55°	57′	23″	286°	7′	44″
Ft. Laramie, Wy. T	42°	12′	10″	27°	44′	42″
Ft. Leavenworth, Kas	39°	21′	14″	17°	51′	49″
Frankfort, Ky	38°	14′	0″	7°	37′	0″
Greenwich	51°	28′	38″	282°	56′	59″
Galveston	29°	18′	17″	17°	43′	58″
Harrisburg	40°	16′	0″	359°	47′	0″
Jefferson City, Mo	38°	36′	0″	15°	5′	0″
Lawrence, Kas	38°	57′	15″	18°	12′	0″
Milwaukie, Wis	43°	2′	24″	10°	51′	3″
Moscow	55°	45′	20″	245°	22′	45″
New York	40°	43′	48″	356°	56′	1″
Paris	48°	50′	11″	280°	36′	47″
Philadelphia	39°	57′	7″	358°	6′	35″
Rome	41°	53′	54″	270°	27′	57″
St. Louis	38°	37′	28″	13°	14′	15″
St. Paul, Minn	44°	52′	46″	18°	1′	53″
St. Petersburg	59°	56′	30″	252°	38′	36″
St. Joseph's, Mo	23°	3′	13″	22°	37′	43″
Washington	38°	53′	39″	00°	00′	00″

The student will find useful exercise in computing the differences in time between any two places named, or between any one place, as New York, and all the others.

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Drawer of a draft.—One who signs an order or directs money to be paid.

Duty.—A tax or customs. A sum of money required by government on the importation, exportation, or consumption of goods.

Exact division.—Division in which the quotient is an exact integral number.

Exchange.—A method of obtaining money in one place, in consideration of money deposited in another place, by means of drafts or bills of exchange.

Exponent.—A number which denotes the order of the power of any number. Usually written in small figures at the right and above the figures of the number whose power is indicated.

Extremes.—The first and last terms of a proportion, or of a series.

Evolution.—The process of finding the root of a number.

Factor.—Any one of two or more numbers, which multiplied together, will produce a given number.

Figure.—A mark or character used as a symbol to represent a number according to the Arabic notation. Also a diagram used to represent a geometrical form.

Fraction.—The expression which represents a fractional number.

Fractional number.—A number which denotes some portion of a thing.

Fractional unit.—*A misnomer.* What is usually so designated is, in fact, a part of a unit.

Geometrical series.—A series in which each number is derived from the preceding one, by multiplying by a constant factor either more or less than one.

Grace.—An allowance of three days after the date mentioned in a note, within which to pay the note.

Gram.—The unit of weight in the metric system, equal to 15.432 troy grains.

Greatest common factor.—The greatest number which is a common factor to several given numbers.

Improper fraction.—Designates a fraction of which the denominator does not exceed the numerator.

Index.—The number which denotes the order of a root indicated. It is written at the left and above the radical sign.

Inland exchange.—Exchange between places in the same country.

Installment.—A payment in part.

Insurance.—A guarantee for partial indemnity for loss of property incurred by fire, storm at sea, or other disaster, or for loss of life.

Integer.—A number which denotes whole things.

Involution.—Process of finding a power of a number.

Latitude.—Distance from the equator of the earth, reckoned in degrees.

Least common multiple.—The least number which is a common multiple of several numbers.

Liability.—A debt.

Line.—Space in length without breadth or thickness.

Liter.—The unit of capacity in the metric system, equal in volume to a cube, of which each edge is a decimeter in length; equivalent to 1.05673 liquid quarts.

Long division.—The method of dividing, in which the necessary processes of multiplication and subtraction are expressed.

Longitude.—Distance between two places on the surface of the earth, reckoned east or west in degrees or in time.

Loss.—In a business transaction is the excess of the cost price above the selling price, or the proceeds of sale.

Maturity of a note.—The date at which a note legally becomes due. At the expiration of three days grace, when those are allowed; otherwise at the date named in the note.

Mean proportional.—A number which appears both as second and third term of a proportion. It follows that the square of a mean proportional is equal to the product of the extremes.

Means.—The terms of a proportion or of a series intervening between the extremes.

Meter.—The unit of length in the metric system; equal to 39.37 inches.

Minuend.—That given number in subtraction which is equal to the sum of another given number called the subtrahend, and a number sought, called the difference or remainder.

Mixed number.—A number expressing both whole things and parts of things counted together.

Multiple.—Is any product of which the given number is one of the integral factors.

Multiplicand.—The number which is to be counted a number of times together.

Multiplication.—The process of counting a number of things a number of times together.

Multiplier.—The number showing how many times the multiplicand is to be counted.

Net proceeds.—What remains of the money received for property after deducting all expenses incurred in disposing of it.

Notation.—A system of expressing numbers, by any characters or symbols.

Note.—A written agreement to pay a specified sum of money at a specified time.

Number.—That which expresses *how many*.

Numeration.—A system of naming numbers.

Order of number.—A name used to designate the number of things in a group or collection of things, as *tens*, *hundreds*, *thousands*, and so on.

Partial payment.—Part payment on a note, which is over-due.

Partnership.—Association of two or more persons, to carry on a business together.

Par value.—Market value equal to cost.

Percentage.—A portion of any given principal reckoned at some rate per cent. Also, the collection of arithmetical processes in which rate per cent is reckoned in the computation.

Period.—A group of three figures.

Policy.—The written contract of insurance.

Premium.—The sum paid for insurance, reckoned as some rate per cent of the sum insured. Also, the excess of market value above par value, and in general any advance in a rate above 100 %.

Present worth.—The present value of a debt which becomes due at some future day. Usually reckoned as that sum which placed at interest would amount to the principal of the debt at the time it becomes due.

Prime number.—Any number which has no integral factors besides itself and one.

Principal.—The sum of money drawing interest.

Problem.—A question, the answer to which becomes apparent by a process of reasoning called a solution.

Product.—The number obtained by multiplication.

Profit.—The excess of selling price, or net proceeds above cost.

Progression.—A series.

Proof.—The evidence which establishes the accuracy of any result.

Proper fraction.—A fraction of which the denominator is greater than the numerator.

Proportion.—An expression of two equal ratios.

Quantity.—That which expresses *how much.*

Rate per cent.—Rate by the hundred.

Rate per unit.—Rate or allowance for one.

Ratio.—The relation expressed by the quotient of one number divided by another.

Reduction.—A changing of form without changing of value.

Remainder.—The difference which added to the subtrahend gives a sum equal to the minuend.

Roman notation.—The system of expressing numbers adopted by the Romans, consisting in the use of seven letters of the alphabet.

Root.—A number which raised to a given power will produce the given number.

Rule.—A prescribed method of procedure.

Security.—Property used to guarantee the payment of any obligation.

Series.—A set of numbers in which any one can be derived from one or more of those that precede according to some rule.

Share.—One of a certain number of portions into which the stock of a company is divided.

Short division.—The method of dividing in which the necessary operations of multiplying and subtracting are performed mentally and not expressed.

Similar fractions.—Fractions having a common denominator.

Solid.—The body which occupies some given volume.

Solution.—The process by means of which the answer to a difficult question is obtained.

Space.—That which contains all bodies.

Square root.—That number which raised to the second power, will produce the given number.

Stock.—The capital invested in business.

Subtraction.—The process of finding the difference between two numbers; or finding the number which added to one of two numbers will produce the other.

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